Multipartite entangled states in particle mixing*

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- Flavor mixing and entanglement;
- Entanglement in neutrino oscillations:
  - Decoherence;
  - Flavor entanglement;
- Neutrino oscillations as a resource for quantum information.
- Particle mixing and entanglement in Quantum Field Theory.

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Motivations

- CKM quark mixing, meson mixing, massive neutrino mixing play a crucial role in phenomenology;

- Evidence of neutrino oscillations;

- Importance of entanglement both at a fundamental level and for quantum information;

- Entanglement in particle physics: entanglement, decoherence, Bell inequalities for the $K^0\bar{K}^0$ (or $B^0\bar{B}^0$) system\(^*\);

- Necessity\(^\dagger\) for a treatment of entanglement in the context of Quantum Field Theory.


Entanglement in particle mixing

- Flavor mixing (neutrinos)

\[
|\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle
\]

\[
|\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle
\]

- Correspondence with two-qubit states:

\[
|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,
\]

where \(|\rangle_i\) denotes states in the Hilbert space for neutrinos with mass \(m_i\).

\Rightarrow\text{ flavor states are entangled superpositions of the mass eigenstates:}

\[
|\nu_e\rangle = \cos \theta |10\rangle + \sin \theta |01\rangle.
\]
Single-particle entanglement*

– A state like $|\psi\rangle_{A,B} = |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B$ is entangled;

– entanglement among field modes, rather than particles;

– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

P.Zanardi, Phys. Rev. A (2002);
J.van Enk, Phys. Rev. A (2005), (2006);
Multipartite entanglement

Characterization of entanglement for multipartite systems is a non-trivial task. Several approaches have been developed: global entanglement, tangle, geometric measures*, etc...

In the 3-qubit case, the two fundamental classes† of states are those of the $GHZ$ state $|GHZ^{(3)}\rangle$ and of the $W$ state $|W^{(3)}\rangle$.

In the $N$-partite instance, such states are defined as:

\[
|GHZ^{(N)}\rangle = \frac{1}{\sqrt{2}}(|000\ldots0\rangle + |111\ldots1\rangle),
\]

\[
|W^{(N)}\rangle = \frac{1}{\sqrt{N}}(|100\ldots0\rangle + |010\ldots0\rangle + |001\ldots0\rangle + \ldots |000\ldots1\rangle).
\]

Multipartite entanglement measures: pure states

Let $\rho = |\psi\rangle\langle\psi|$ be the density operator corresponding to a pure state $|\psi\rangle$, describing the system $S$ partitioned into $N$ parties.

Bipartition of the $N$-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; \quad (1 \leq n < N)$$

and

$$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\}, \quad 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; \quad i_q \neq j_p$$

- Reduced density matrix of $S_{A_n}$ after tracing over $S_{B_{N-n}}$:

$$\rho_{A_n} \equiv \rho_{i_1,i_2,\ldots,i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1,j_2,\ldots,j_{N-n}}[\rho]$$

• von Neumann entropy associated with the above bipartition:

\[ E_{vN}^{(A_n;B_{N-n})} = -Tr_{A_n} [\rho_{A_n} \log_2 \rho_{A_n}] . \]

• Average von Neumann entropy (global entanglement)

\[ \langle E_{vN}^{(n:N-n)} \rangle = \binom{N}{n}^{-1} \sum_{A_n} E_{vN}^{(A_n;B_{N-n})} , \]

where the sum is intended over all the possible bipartitions of the system in two subsystems each with \( n \) and \( N-n \) elements \((1 \leq n < N)\).
Examples

3-qubits
only unbalanced bipartitions \((S_{A_2}, S_{B_1})\) of two subsystems can be considered:

\[ E_{21}^{(3)} \equiv E_{vN}^{(A_2; B_1)}(\rho_{W(3)}) = \langle E_{vN}^{(2:1)}(\rho_{W(3)}) \rangle = \log_2 3 - \frac{2}{3} \approx 0.918296, \]

\[ E_{vN}^{(A_2; B_1)}(\rho_{GHZ(3)}) = \langle E_{vN}^{(2:1)}(\rho_{GHZ(3)}) \rangle = 1. \]

4-qubits
both unbalanced, i.e. \((S_{A_3}, S_{B_1})\), and balanced bipartitions, i.e. \((S_{A_2}, S_{B_2})\) can be considered:

\[ E_{31}^{(4)} \equiv E_{vN}^{(A_3; B_1)}(\rho_{W(4)}) = \langle E_{vN}^{(3:1)}(\rho_{W(4)}) \rangle = 2 - \frac{3}{4} \log_2 3 \approx 0.811278, \]

\[ E_{22}^{(4)} \equiv E_{vN}^{(A_2; B_2)}(\rho_{W(4)}) = \langle E_{vN}^{(2:2)}(\rho_{W(4)}) \rangle = 1. \]
Multipartite entanglement measures: mixed states

– Entropic measures cannot be used to quantify the entanglement of mixed states ⇒ logarithmic negativity.

We denote by

$$\tilde{\rho}_{A_n} \equiv \rho^{PTB_{N-n}} = \rho^{PTj_1,j_2,\ldots,j_{N-n}}$$

the bona fide density matrix, obtained by the partial transposition of $\rho$ with respect to the parties belonging to the subsystem $S_{B_{N-n}}$.

• Logarithmic negativity associated with the above bipartition

$$E_N^{(A_n;B_{N-n})} = \log_2 \| \tilde{\rho}_{A_n} \|_1 .$$

• Average logarithmic negativity (global entanglement)

$$\langle E_N^{(n:N-n)} \rangle = \left( \begin{array}{c} N \\ n \end{array} \right)^{-1} \sum_{A_n} E_N^{(A_n;B_{N-n})} ,$$

where the sum is intended over all the possible bipartitions of the system.
Multipartite entanglement in neutrino mixing

– Neutrino mixing (three flavors):

$$\left| \nu_f \right\rangle = U(\bar{\theta}, \delta) \left| \nu_m \right\rangle$$

with $$\left| \nu_f \right\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$$ and $$\left| \nu_m \right\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$$.

– Mixing matrix (MNSP)

$$U(\bar{\theta}, \delta) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{13}e^{i\delta} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}s_{13} \end{pmatrix},$$

where $$(\bar{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta), c_{ij} \equiv \cos \theta_{ij}$$ and $$s_{ij} \equiv \sin \theta_{ij}$$.

• Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

Flavor states as generalized W states

– Define the generalized class of three-qubit W states as

\[ |W^{(3)}(\tilde{\theta}; \delta)\rangle \equiv U^{(3f)}(\tilde{\theta}, \delta) |\nu^{(3)}\rangle \]

\[ U^{(3f)}(\tilde{\theta}, \delta) = U(\tilde{\theta}, \delta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}, \]

where \[ |W^{(3)}(\tilde{\theta}; \delta)\rangle = (|W_e^{(3)}(\tilde{\theta}, \delta)\rangle, |W_\mu^{(3)}(\tilde{\theta}, \delta)\rangle, |W_\tau^{(3)}(\tilde{\theta}, \delta)\rangle)^T \]
and \[ |\nu^{(3)}\rangle = (|\nu_1^{(3)}\rangle, |\nu_2^{(3)}\rangle, |\nu_3^{(3)}\rangle)^T. \]

– The entanglement properties of the states associated with matrices \( U(\tilde{\theta}, \delta) \)
and \( U^{(3f)}(\tilde{\theta}, \delta) \) are identical.

\[ \Leftrightarrow \text{we regard flavor neutrino states as generalized W states.} \]
Entanglement properties of states with maximal mixing

- Flavor mixing is maximal for

\[ \theta_{12}^{\text{max}} = \frac{\pi}{4}; \quad \theta_{23}^{\text{max}} = \frac{\pi}{4}; \quad \theta_{13}^{\text{max}} = \arccos \sqrt{\frac{2}{3}}; \quad \delta^{\text{max}} = \frac{\pi}{2}. \]

⇒ all elements of CKM matrix have modulus = 1/3:

\[ U_{\text{max}}^{(3f)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ iy & iy^2 & i \\ iy^2 & iy & i \end{pmatrix} \quad \text{with} \quad y = \exp \left( 2i\pi/3 \right). \]

In this case, all the \(|W_\alpha^{(3)}(\tilde{\theta}, \delta)\rangle\) states have the same entanglement of \(|W^{(3)}\rangle\):

\[ E_{vN}^{(2;1)}(|W^{(3)}(\tilde{\theta}_{\text{max}}, \delta^{\text{max}})\rangle) = \langle E_{vN}^{(2;1)}(|W^{(3)}(\tilde{\theta}_{\text{max}}, \delta^{\text{max}})\rangle) \rangle = E_{21}^{(3)}. \]
Entanglement properties of states $|W_\alpha^{(3)}(\tilde{\theta}^{\text{max}}; \delta)\rangle$ ($\alpha = e, \mu, \tau$)

We study the dependence of entanglement on the phase $\delta$, with the rotation angles set at their maximal values $\theta_{ij}^{\text{max}}$.

The matrix $U^{(3f)}$ becomes

$$U^{(3f)}(\delta) = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
-\frac{1}{2}(\sqrt{3} + e^{i\delta}) & \frac{1}{2}(\sqrt{3} - e^{i\delta}) & e^{i\delta} \\
\frac{1}{2}(\sqrt{3} - e^{i\delta}) & -\frac{1}{2}(\sqrt{3} + e^{i\delta}) & e^{i\delta}
\end{pmatrix}.$$  

We get:

$$E_{vNe}^{(1,2;3)} = E_{vNe}^{(1,3;2)} = E_{vNe}^{(2,3;1)} = E_{vN\mu}^{(1,2;3)} = E_{vN\tau}^{(1,2;3)} = \log_2 3 - \frac{2}{3},$$

$$E_{vN\mu}^{(1,3;2)} = E_{vN\tau}^{(2,3;1)} = -\left(\frac{1}{3} - \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{1}{3} - \frac{\cos \delta}{2\sqrt{3}}\right] - \left(\frac{2}{3} + \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{2}{3} + \frac{\cos \delta}{2\sqrt{3}}\right],$$

$$E_{vN\mu}^{(2,3;1)} = E_{vN\tau}^{(1,3;2)} = -\left(\frac{1}{3} + \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{1}{3} + \frac{\cos \delta}{2\sqrt{3}}\right] - \left(\frac{2}{3} - \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{2}{3} - \frac{\cos \delta}{2\sqrt{3}}\right],$$

where $E_{vN\alpha}^{(i,j;k)} = E_{vN}^{(i,j;k)}(|W_\alpha^{(3)}(\delta)\rangle).$
Entanglement of the state $|W^{(3)}_{\mu} (\tilde{\theta}_{\text{max}}; \delta)\rangle$

von Neumann entropy $E^{(i,j;k)}_{vN \mu}$ and average von Neumann entropy $\langle E^{(2:1)}_{vN \mu}\rangle$ (full line) as functions of the $CP$-violating phase $\delta$.

$E^{(i,j;k)}_{vN \mu}$ is plotted for the bipartitions $i = 1, j = 2, k = 3$ (dotted line); $i = 1, j = 3, k = 2$ (dashed line); $i = 2, j = 3, k = 1$ (dot-dashed line).

$E^{(1,2;3)}_{vN \mu}$ is constant and takes the reference value $E^{(3)}_{21} = 0.918296$. 
• “Squeezing” of entanglement: $E_{vN\mu}^{(1,3;2)}$ and $E_{vN\mu}^{(2,3;1)}$ vary with $\delta$, attaining the absolute maximum 1 at the points $\delta_1 = \pm \arccos\left(-\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ and $\delta_2 = \pm \arccos\left(\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ (with $p$ integer), respectively, and exceeding the reference value $E_{21}^{(3)}$.

The average von Neumann entropy $\langle E_{vN\mu}^{(2;1)} \rangle$ stays below the reference value $E_{21}^{(3)}$, attaining it at the points $\delta = \frac{\pi}{2} \pm p\pi$.

• The free parameter $\delta$ can be used to concentrate and squeeze the entanglement in a specific bipartition, allowing a sharply peaked distribution of entanglement, at the expense of the average von Neumann entropy.
Quantifying entanglement in quark and neutrino flavor mixing

– For quarks, the parameters in the CKM matrix take the values:
\[ \theta_{12}^{\text{CKM}} = 13.0^\circ \pm 0.1^\circ, \quad \theta_{13}^{\text{CKM}} = 0.2^\circ \pm 0.1^\circ, \quad \theta_{23}^{\text{CKM}} = 2.4^\circ \pm 0.1^\circ, \quad \delta^{\text{CKM}} = 1.05 \pm 0.24. \]

Correspondingly, the von Neumann entropies are

\[
\begin{array}{|c|c|c|c|c|}
\hline
\alpha & E_{vN\alpha}^{(d,s;b)} & E_{vN\alpha}^{(d,b;s)} & E_{vN\alpha}^{(s,b;d)} & \langle E_{vN\alpha}^{(2:1)} \rangle \\
\hline
d' & 0.0002 & 0.2889 & 0.2890 & 0.1927 \\
s' & 0.0185 & 0.2960 & 0.2887 & 0.2011 \\
b' & 0.0186 & 0.0180 & 0.0010 & 0.0126 \\
\hline
\end{array}
\]

v.N. entropies \( E_{vN\alpha}^{(i,j;k)} \) and \( \langle E_{vN\alpha}^{(2:1)} \rangle \) (\( \alpha = d', s', b' \)) for the three-flavor states associated with the quark mixing.

– Entanglement stays low, and it concentrates in the bipartitions \((d, b; s)\) and \((s, b; d)\) of the states \(|d'\rangle\) and \(|s'\rangle\), while it is very small for the state \(|b'\rangle\).
– For neutrinos, recent estimates of elements of the MNSP matrix are

\[
\sin^2 \theta_{12}^{MNSP} = 0.314(1 + 0.18 - 0.15), \quad \sin^2 \theta_{13}^{MNSP} = (0.8 + 2.3 - 0.8) \times 10^{-2}, \quad \sin^2 \theta_{23}^{MNSP} = 0.45(1 + 0.35 - 0.20)
\]

– The $CP$-violating phase associated with lepton mixing is still undetermined; therefore, $\delta^{MNSP}$ may take an arbitrary value in the interval $[0, 2\pi)$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$E_{vN\alpha}^{(1,2;3)}$</th>
<th>$E_{vN\alpha}^{(1,3;2)}$</th>
<th>$E_{vN\alpha}^{(2,3;1)}$</th>
<th>$\langle E_{vN\alpha}^{(2:1)} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>0.0672</td>
<td>0.8948</td>
<td>0.9038</td>
<td>0.5995</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.9916</td>
<td>0.9220 - 0.9813</td>
<td>0.5679 - 0.7536</td>
<td>0.8469 - 0.8891</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.9939</td>
<td>0.8397 - 0.9352</td>
<td>0.4784 - 0.6922</td>
<td>0.8025 - 0.8419</td>
</tr>
</tbody>
</table>

v.N. entropies corresponding to the neutrino flavor states. The given intervals of possible values are due to the freedom in the choice of the $CP$-violating phase.

v.N. entropies $E_{vN\mu}^{(i,j;k)}$ and average v.N. entropy $\langle E_{vN\mu}^{(2;1)} \rangle$ as functions of the $CP$-violating phase $\delta$. The mixing angles $\theta_{ij}^{MNSP}$ are assumed to be Gaussian random variables, distributed around the mean values $\bar{\theta}_{ij}^{MNSP}$ coinciding with the experimental values.
Neutrino oscillations (plane waves)*

\[ |\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle \]
\[ |\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle \]

- **Time evolution:**

\[ |\nu_e(t)\rangle = \cos \theta e^{-iE_1 t} |\nu_1\rangle + \sin \theta e^{-iE_2 t} |\nu_2\rangle \]

- **Flavor oscillations:**

\[ P_{\nu_e \rightarrow \nu_e}(t) = |\langle \nu_e | \nu_e(t) \rangle|^2 = 1 - \sin^2 2\theta \sin^2 \left( \frac{\Delta E}{2} t \right) = 1 - P_{\nu_e \rightarrow \nu_\mu}(t) \]

- **Flavor conservation:**

\[ |\langle \nu_e | \nu_e(t) \rangle|^2 + |\langle \nu_\mu | \nu_e(t) \rangle|^2 = 1 \]

- The entanglement of \( |\nu_e(t)\rangle \) in terms of the qubits \( |\nu_1\rangle, |\nu_2\rangle \) does not depend on time.

Neutrino oscillations (wave packets)*

– Consider, in one dimension, a neutrino with definite flavor, propagating along the $x$ direction:

$$|\nu_\alpha(x, t)\rangle = \sum_j U_{\alpha,j} \psi_j(x, t) |\nu_j\rangle,$$

where $U_{\alpha,j}$ is an element of the mixing matrix, $|\nu_j\rangle$ the mass eigenstate with mass $m_j$, and $\psi_j(x, t)$ its wave function.

– Assume Gaussian distribution $\psi_j(p)$ for the momentum of the massive neutrino $|\nu_j\rangle$:

$$\psi_j(x, t) = \frac{1}{\sqrt{2\pi}} \int dp \psi_j(p) e^{ipx-iE_j(p)t}, \quad \psi_j(p) = \frac{1}{(2\pi \sigma_p^2)^{1/4}} e^{-\frac{1}{4\sigma_p^2}(p-p_j)^2},$$

where $E_j(p) = \sqrt{p^2 + m_j^2}$.

– The associated density matrix writes:

$$\rho_\alpha(x, t) = |\nu_\alpha(x, t)\rangle\langle\nu_\alpha(x, t)|.$$

If $\sigma_p \ll E_j^2(p_j)/m_j$, one can write $E_j(p) \simeq E_j + v_j(p-p_j)$, with $E_j \equiv \sqrt{p_j^2 + m_j^2}$, and $v_j \equiv \frac{\partial E_j(p)}{\partial p} |_{p=p_j} = \frac{p_j}{E_j}$ is the group velocity of the wave packet for $\nu_j$.

– In this case, a Gaussian integration yields:

\[
\rho_\alpha(x, t) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \sum_{j,k} U_{\alpha j} U_{\alpha k}^* e^{-i(E_j - E_k)t + i(p_j - p_k)x - \frac{1}{\sigma_x^2}[(x-v_j t)^2 + (x-v_k t)^2]} |\nu_j\rangle \langle \nu_k|,
\]

where \( \sigma_x = (2\sigma_p)^{-1} \). For extremely relativistic neutrinos, one has

\[
E_j \simeq E + \xi \frac{m_j^2}{2E}, \quad p_j \simeq E - (1 - \xi) \frac{m_j^2}{2E}, \quad v_j \simeq 1 - \frac{m_j^2}{2E_j^2}
\]

where \( E \) is the neutrino energy in the limit of zero mass, and \( \xi \) a dimensionless constant depending on the characteristic of the production process.

– The density matrix \( \rho_\alpha(x, t) \) provides a space-time description of neutrino dynamics.

– In realistic situations, it is convenient to consider the time-independent density matrix \( \rho_\alpha(x) \) obtained by the time average of \( \rho_\alpha(x, t) \):

\[
\rho_\alpha(x) = \sum_{j,k} U_{\alpha j} U_{\alpha k}^* \exp \left[-i \frac{\Delta m_{jk}^2 x}{2E} - \left( \frac{\Delta m_{jk}^2 x}{4\sqrt{2E^2\sigma_x}} \right)^2 - \left( \xi \frac{\Delta m_{jk}^2}{4\sqrt{2E\sigma_p}} \right)^2 \right] |\nu_j\rangle \langle \nu_k|,
\]

with \( \Delta m_{jk}^2 = m_j^2 - m_k^2 \).
Flavor oscillations in space.

\[ P_{\nu_e \rightarrow \nu_e}(x) \simeq 1 - \frac{1}{2} \sin^2(2\theta) \left\{ 1 - \cos \left( 2\pi \frac{x}{L_{\text{osc}}} \right) \exp \left[ - \left( \frac{x}{L_{\text{coh}}} \right)^2 - 2\pi^2 \left( \frac{\sigma_x}{L_{\text{osc}}} \right)^2 \right] \right\} \]

- Oscillation length: \( L_{\text{osc}} = \frac{4\pi p}{\Delta m^2} \)

- Coherence length: \( L_{\text{coh}} = (L_{\text{osc}} p) / (\sqrt{2\pi} \sigma_p) \).
Decoherence in neutrino oscillations

- We analyze the coherence of the quantum superposition of the neutrino mass eigenstates, by looking at the spatial behavior of the multipartite entanglement of the above state\(^*\).

By means of the identification \(|\nu_i\rangle = |\delta_{i,1}\rangle_1|\delta_{i,2}\rangle_2|\delta_{i,3}\rangle_3 \equiv |\delta_{i,1}\delta_{i,2}\delta_{i,3}\rangle\), with \(i = 1, 2, 3\), we construct the matrix with elements

\[
\langle lmn|\rho_\alpha(x)|ijk\rangle, \quad \text{where} \quad i, j, k, l, m, n = 0, 1
\]

- We analytically compute logarithmic negativities \(E_{N_\alpha}^{(i,j;k)}\), for \(i, j, k = 1, 2, 3\) and \(i \neq j \neq k\), and average logarithmic negativity \(\langle E_{N_\alpha}^{(2:1)}\rangle\), for the neutrino states with flavor \(\alpha = e, \mu, \tau\).

We assume for the mixing angles the experimental values
\[ \sin^2 \theta^{MNSP}_{12} = 0.314(1^{+0.18}_{-0.15}) , \quad \sin^2 \theta^{MNSP}_{13} = (0.8^{+2.3}_{-0.8}) \times 10^{-2} , \quad \sin^2 \theta^{MNSP}_{23} = 0.45(1^{+0.35}_{-0.20}) \]

The squared mass differences are fixed at the experimental values*
\[
\Delta m^2_{21} = \delta m^2 , \quad \Delta m^2_{31} = \Delta m^2 + \frac{\delta m^2}{2} , \quad \Delta m^2_{32} = \Delta m^2 - \frac{\delta m^2}{2} , \\
\delta m^2 = 7.92 \times 10^{-5} eV^2 , \quad \delta m^2 = 2.6 \times 10^{-3} eV^2 .
\]

We take \( E = 10 GeV \) and \( \sigma_p = 1 GeV \). The parameter \( \xi \) is put to zero for simplicity.

Logarithmic negativities $E_{Ne}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{Ne}^{(2:1)} \rangle$ (solid line) as functions of the distance $x$ (meters).

In panel II we plot a zoom of $E_{Ne}^{(1,3;2)}$ and $E_{Ne}^{(2,3;1)}$.

All plotted quantities are independent of the CP-violating phase $\delta$. 
Logarithmic negativities $E_{\mathcal{N}\mu}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{\mathcal{N}\alpha}^{(2:1)} \rangle$ (solid line), with $\alpha = \mu, \tau$, as functions of the distance $x$ (meters).

The CP-violating phase $\delta$ is put to zero. The $x$ axis is in logarithmic scale, and the dimensions are meters.
Logarithmic negativities $E_{\mathcal{N}_\mu}^{(1,3;2)}$ (panel I) and $E_{\mathcal{N}_\mu}^{(2,3;1)}$ (panel II) as functions of the distance $x$ (meters) for different choices of the CP-violating phase $\delta$: (a) $\delta = 0$ (dotted line); (b) $\delta = \frac{\pi}{2}$ (dashed line); (b) $\delta = \pi$ (dot-dashed line). $E_{\mathcal{N}_\mu}^{(1,2;3)}$ is independent of $\delta$. 
(Flavor) Entanglement in neutrino oscillations

– Two-flavor neutrino states

\[ |\nu^{(f)}\rangle = U(\tilde{\theta}, \delta) |\nu^{(m)}\rangle \]

where \( |\nu^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T \) and \( |\nu^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T \) and \( U(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \).

– Flavor states at time \( t \):

\[ |\nu^{(f)}(t)\rangle = U(\tilde{\theta}, \delta) U_0(t) U(\tilde{\theta}, \delta)^{-1} |\nu^{(f)}\rangle \equiv \tilde{U}(t) |\nu^{(f)}\rangle, \]

with \( U_0(t) = \begin{pmatrix} e^{-iE_1t} & 0 \\ 0 & e^{-iE_2t} \end{pmatrix} \).

– Transition probability for $\nu_\alpha \rightarrow \nu_\beta$

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = |\langle \nu_\beta|\nu_\alpha(t)\rangle|^2 = |\tilde{U}_{\alpha\beta}(t)|^2.$$ 

• We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu \equiv |01\rangle_f,$$

– Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle = \tilde{U}_{\alpha e}(t)|1\rangle_e|0\rangle_\mu + \tilde{U}_{\alpha \mu}(t)|0\rangle_e|1\rangle_\mu, \quad \alpha = e, \mu.$$
(Flavor) Entanglement in neutrino oscillations: three flavors

Three-flavor neutrino states

$$|\nu^{(f)}\rangle = U(\tilde{\theta}, \delta) |\nu^{(m)}\rangle$$

where $$|\nu^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$$ and $$|\nu^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$$

$$U(\tilde{\theta}, \delta) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13} e^{i\delta} & s_{12}s_{23} - s_{12}c_{23}s_{13} e^{i\delta} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13} e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$,

where $$(\tilde{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta), c_{ij} \equiv \cos \theta_{ij} \text{ and } s_{ij} \equiv \sin \theta_{ij}.$$
– Flavor states at time $t$:

$$|\nu_f(t)\rangle = U(\tilde{\theta}, \delta) U_0(t) U(\tilde{\theta}, \delta)^{-1} |\nu_f\rangle \equiv \tilde{U}(t)|\nu_f\rangle,$$

with $|\nu_f\rangle$ flavor states at $t = 0$, $U_0(t) = \text{diag}(e^{-iE_1t}, e^{-iE_2t}, e^{-iE_3t})$,

and $\tilde{U}(t) = U(\tilde{\theta}, \delta) U_0(t) U(\tilde{\theta}, \delta)^{-1}$, with $\tilde{U}(t = 0) = \mathbb{I}$.

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\tilde{U}_{\alpha\beta}(t)|^2,$$

• Take the flavor states at time $t = 0$ as the qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu |0\rangle_\tau \equiv |100\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu |0\rangle_\tau \equiv |010\rangle_f,$$

$$|\nu_\tau\rangle \equiv |0\rangle_e |0\rangle_\mu |1\rangle_\tau \equiv |001\rangle_f$$
Entanglement measure

– Let $\rho = |\psi\rangle\langle\psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the $N$-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}$, $1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$

and

$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\}$, $1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$

– Reduced density matrix of $S_{A_n}$ after tracing over $S_{B_{N-n}}$:

$\rho_{A_n} \equiv \rho_{i_1,i_2,\ldots,i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1,j_2,\ldots,j_{N-n}}[\rho]$
• Linear entropy associated to such a bipartition:

\[ S_L^{(A_n; B_{N-n})}(\rho) = \frac{d}{d-1} \left( 1 - Tr_{A_n}[\rho_{A_n}^2] \right), \]

d is the Hilbert-space dimension: \( d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}. \)

• Average linear entropy (global entanglement):

\[ \langle S_L^{(n:N-n)}(\rho) \rangle = \binom{N}{n}^{-1} \sum_{A_n} S_L^{(A_n; B_{N-n})}(\rho), \]

sum over all the possible bi-partitions of the system in two subsystems, respectively with \( n \) and \( N - n \) elements (\( 1 \leq n < N \)).
Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu \Rightarrow \rho^{(e)}$.

– The associated linear entropy is:

$$S_L^{(e;\mu)}(\rho^{(e)}) = 4 |\tilde{U}_{e\mu}(t)|^2 |\tilde{U}_{ee}(t)|^2 = 4 P_{\nu_e\rightarrow\nu_e(t)} P_{\nu_e\rightarrow\nu_\mu(t)}$$

– The linear entropy for the state $\rho^{(\alpha)}$ is:

$$S_L^{(e;\mu)} = S_L^{(\mu;e)} = \langle S_L^{(1:1)} \rangle = 4 |\tilde{U}_{\alpha\mu}(t)|^2 |\tilde{U}_{\alpha e}(t)|^2$$

$$= 4 |\tilde{U}_{\alpha e}(t)|^2 (1 - |\tilde{U}_{\alpha e}(t)|^2)$$

$$= 4 |\tilde{U}_{\alpha\mu}(t)|^2 (1 - |\tilde{U}_{\alpha\mu}(t)|^2).$$

• Linear entropy given by product of transition probabilities!
Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2 \theta = 0.314$. Transition probabilities $P_{\nu_e \rightarrow \nu_e}$ (dashed) and $P_{\nu_e \rightarrow \nu_\mu}$ (dot-dashed) are reported for comparison.
Entanglement in neutrino oscillations: three-flavors

– In the three-flavor case, we obtain

\[ S_{L\alpha}^{(e,\mu;\tau)} = 4|\tilde{U}_{\alpha\tau}(t)|^2 (|\tilde{U}_{\alpha e}(t)|^2 + |\tilde{U}_{\alpha\mu}(t)|^2) \]

\[ = 4|\tilde{U}_{\alpha\tau}(t)|^2 (1 - |\tilde{U}_{\alpha\tau}(t)|^2). \]

The linear entropies for the two remaining bi-partitions are easily obtained by permuting the indexes \( e, \mu, \tau \).

– The average linear entropy is

\[ \langle S_{L\alpha}^{(2:1)} \rangle = \frac{8}{3} (|\tilde{U}_{\alpha e}(t)|^2 |\tilde{U}_{\alpha\mu}(t)|^2 + |\tilde{U}_{\alpha e}(t)|^2 |\tilde{U}_{\alpha\tau}(t)|^2 + |\tilde{U}_{\alpha\mu}(t)|^2 |\tilde{U}_{\alpha\tau}(t)|^2). \]
Linear entropies $S_{Le}^{(\alpha,\beta;\gamma)}$ and $\langle S_{Le}^{(2;1)} \rangle$ as functions of the scaled time $T = \frac{2E_t}{\Delta m_{12}^2}$. Curves correspond to the partial linear entropies $S_{Le}^{(e,\mu;\tau)}$ (long-dashed), $S_{Le}^{(e,\tau;\mu)}$ (dashed), $S_{Le}^{(\mu,\tau;e)}$ (dot-dashed), and to the average linear entropy $\langle S_{Le}^{(2;1)} \rangle$ (full).

Parameters are fixed at central experimental values: $\sin^2 \theta_{12} = 0.314$, $\sin^2 \theta_{23} = 0.45$, $\sin^2 \theta_{12} = 0.008$, $\Delta m_{12}^2 = 7.92 \times 10^{-5} eV^2$, $\Delta m_{23}^2 = 2.6 \times 10^{-3} eV^2$. 

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Because of \( CPT \) invariance, the \( CP \) asymmetry \( \Delta_{CP}^{\alpha,\beta} \) is equal to the asymmetry under time reversal:

\[
\Delta_{CP}^{\alpha,\beta} = \Delta_T^{\alpha,\beta} = P_{\nu\alpha \rightarrow \nu\beta}(t) - P_{\nu\beta \rightarrow \nu\alpha}(t) = P_{\nu\alpha \rightarrow \nu\beta}(t) - P_{\nu\alpha \rightarrow \nu\beta}(-t).
\]

\( \Delta_{CP}^{\alpha,\beta} \neq 0 \) for \( \delta \neq 0 \). Note that \( \sum_{\beta} \Delta_{CP}^{\alpha\beta} = 0 \) with \( \alpha, \beta = e, \mu, \tau \).

- Define the "imbalances", i.e. the difference between the linear entropies and their time-reversed expressions:

\[
\Delta S_{L\lambda}^{(\alpha,\beta;\gamma)} = S_{L\lambda}^{(\alpha,\beta;\gamma)}(t) - S_{L\lambda}^{(\alpha,\beta;\gamma)}(-t),
\]

- We have for example:

\[
\Delta S_{Le}^{(e,\mu;\tau)} = 4 \Delta_{CP}^{e,\mu}(|\tilde{U}_{e\tau}(t)|^2 + |\tilde{U}_{\tau e}(t)|^2 - 1),
\]

where the last factor is \( CP \)-even.
The imbalances $\Delta S_{Le}^{(\alpha, \beta; \gamma)}$ as functions of the scaled time $T$. Curves correspond to $\Delta S_{Le}^{(e, \mu; \tau)}$ (long-dashed) and $\Delta S_{Le}^{(e, \tau; \mu)}$ (dot-dashed). The quantity $\Delta S_{Le}^{(\mu, \tau; e)}$ is vanishing.

The $CP$-violating phase is set at the value $\delta = \pi/2$. 
Neutrino oscillations as a resource for quantum information

- Single-particle entanglement encoded in flavor states \( |\nu^{(f)}(t)\rangle \) is a real physical resource that can be used, at least in principle, for protocols of quantum information.

- Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes*.

Charged-current interaction between a neutrino \( \nu_\alpha \) with flavor \( \alpha \) and a nucleon \( N \) gives a lepton \( \alpha^- \) and a baryon \( X \):

\[
\nu_\alpha + N \longrightarrow \alpha^- + X.
\]
Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \rightarrow \alpha^- + p$ with $\alpha = e, \mu$.

A spatially nonuniform magnetic field $B(r)$ constrains the momentum of the outgoing lepton within a solid angle $\Omega_i$, and ensures spatial separation between lepton paths.

The reaction produces a superposition of electronic and muonic spatially separated states.
Given the initial Bell-like superposition \(|\nu_\alpha(t)\rangle\) the unitary process associated with the weak interaction leads to the superposition

\[ |\alpha(t)\rangle = \Lambda_e |1_e\rangle|0_\mu\rangle + \Lambda_\mu |0_e\rangle|1_\mu\rangle, \]

where \(|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1\), and \(|k\rangle_\alpha\), with \(k = 0, 1\), represents the lepton qubit.

The coefficients \(\Lambda_\alpha\) are proportional to \(\tilde{U}_{\alpha\beta}(t)\) and to the cross sections associated with the creation of an electron or a muon.

Analogy with single-photon system: quantum uncertainty on the so-called “which path” of the photon at the output of an unbalanced beam splitter \(\Leftrightarrow\) uncertainty on the “which flavor” of the produced lepton.

The coefficients \(\Lambda_\alpha\) plays the role of the transmissivity and of the reflectivity of the beam splitter.
Entanglement for mixed particles in QFT

- Extension of the above analysis to QFT
- Non-trivial nature of mixing transformations in QFT
- Dynamical symmetry approach to entanglement

*M.Blasone, F. Dell’Anno and S.De Siena, work in progress.*
Dynamical symmetry approach to entanglement

- Entanglement can be characterized by total variance of the operators generating the dynamical algebra*.

- Consider the observables $X_i$ elements of the basis of a Lie algebra $\mathcal{L}$ such that the Lie group $G = \exp(i\mathcal{L})$ defines the dynamic symmetry of the system.

Entanglement of a state $\psi$ of the system is given by the total amount of uncertainty:

$$\Delta(\psi) = \sum_i \left( \langle \psi | X_i | \psi \rangle - \langle \psi | X_i | \psi \rangle^2 \right)$$

– Define neutrino states with definite masses as:

\[ |\nu_i\rangle \equiv \alpha_i^\dagger |0\rangle_m, \quad i = 1, 2 \]

where \( \alpha_i \) is the (fermionic) annihilation operator for a neutrino with mass \( m_i \) and \( |0\rangle_m \equiv |0\rangle_1 \otimes |0\rangle_2 \).

– Flavor annihilation operators:

\[
\begin{align*}
\alpha_e(t) &= \cos \theta \alpha_1(t) + \sin \theta \alpha_2(t) \\
\alpha_\mu(t) &= -\sin \theta \alpha_2(t) + \cos \theta \alpha_1(t)
\end{align*}
\]

where \( \alpha_i(t) = e^{i\omega_i t} \alpha_i \).

– Flavor states:

\[ |\nu_\sigma(t)\rangle \equiv \alpha_\sigma^\dagger(t) |0\rangle_m, \quad \sigma = e, \mu. \]
– Flavor oscillations can be seen equivalently in terms of expectation values of number operators $N_\sigma(t) = \alpha_\sigma(t)\dagger\alpha_\sigma(t)$:

$$P_{\nu_e\rightarrow\nu_e}(t) = |\langle \nu_e | \nu_e(t) \rangle|^2 = \langle \nu_e | N_e(t) | \nu_e \rangle$$

$$P_{\nu_e\rightarrow\nu_\mu}(t) = |\langle \nu_\mu | \nu_e(t) \rangle|^2 = \langle \nu_e | N_\mu(t) | \nu_e \rangle$$

- Variance of the number operators $N_i$ and $N_\sigma(t)$:

– Variance of $N_i \Rightarrow$ static entanglement:

$$\Delta N_i(\nu_e) = \cos^2 \theta \sin^2 \theta$$

– Variance of $N_\sigma(t) \Rightarrow$ flavor entanglement:

$$\Delta N_\sigma(\nu_e)(t) = P_{\nu_e\rightarrow\nu_e}(t) P_{\nu_e\rightarrow\nu_\mu}(t)$$

Both results are proportional to the respective quantities obtained by means of the linear entropy.
Quantum field theory of fermion mixing

Consider mixing relations for two Dirac fields

\[
\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta
\]

\[
\nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta
\]

\(\nu_1, \nu_2\) are fields with definite masses.

The above mixing transformations connect the two quadratic forms:

\[
\mathcal{L} = \bar{\nu}_1 (i \nabla - m_1) \nu_1 + \bar{\nu}_2 (i \nabla - m_2) \nu_2
\]

and

\[
\mathcal{L} = \bar{\nu}_e (i \nabla - m_e) \nu_e + \bar{\nu}_\mu (i \nabla - m_\mu) \nu_\mu - m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e)
\]

with \(m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta, m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta, m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta\).
Currents and charges for mixed fermions

- Lagrangian in the mass basis:

\[ \mathcal{L} = \bar{\Psi}_m (i \partial - M_d) \Psi_m \]

where \( \Psi^T_m = (\nu_1, \nu_2) \) and \( M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \).

- \( \mathcal{L} \) invariant under global \( U(1) \) with conserved (Noether) charge \( Q = \) total charge.

– Consider now the \( SU(2) \) transformation:

\[ \Psi'_m = e^{i\alpha_j \tau_j} \Psi_m ; \quad j = 1, 2, 3. \]

with \( \tau_j = \sigma_j / 2 \) and \( \sigma_j \) being the Pauli matrices.

The associated currents are:

$$
\delta \mathcal{L} = i \alpha_j \bar{\Psi}_m [\tau_j, M_d] \psi_m = -\alpha_j \partial_\mu J_{m,j}^\mu \\
J_{m,j}^\mu = \bar{\Psi}_m \gamma^\mu \tau_j \psi_m
$$

– The charges $Q_{m,j}(t) \equiv \int d^3x J_{m,j}^0(x)$, satisfy the $su(2)$ algebra:

$$
[Q_{m,j}(t), Q_{m,k}(t)] = i \epsilon_{jkl} Q_{m,l}(t).
$$

– The Casimir operator is proportional to the total charge: $C_m = \frac{1}{2} Q$.

• $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for $\nu_1$ and $\nu_2$:

$$
Q_{\nu_1} = \frac{1}{2} Q + Q_{m,3} \\
Q_{\nu_2} = \frac{1}{2} Q - Q_{m,3}
$$

so they can be identified with the flavor charges in the absence of mixing.
The currents in the flavor basis

– Lagrangian in the flavor basis:

\[ \mathcal{L} = \bar{\Psi}_f (i \not{\partial} - M) \Psi_f \]

where \( \Psi_f^T = (\nu_e, \nu_\mu) \) and \( M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix} \).

– Consider the \( SU(2) \) transformation:

\[ \Psi'_f = e^{i\alpha_j \tau_j} \Psi_f \quad ; \quad j = 1, 2, 3. \]

with \( \tau_j = \sigma_j/2 \) and \( \sigma_j \) being the Pauli matrices.

– The charges \( Q_{f,j} \equiv \int d^3x J^0_{f,j} \) satisfy the \( su(2) \) algebra:

\[ [Q_{f,j}(t), Q_{f,k}(t)] = i \epsilon_{jkl} Q_{f,l}(t). \]

– The Casimir operator is proportional to the total charge \( C_f = C_m = \frac{1}{2} Q \).
• $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between $\nu_e$ and $\nu_\mu$.

Define the flavor charges as:

$$Q_{\nu_e}(t) \equiv \frac{1}{2}Q + Q_{f,3}(t)$$

$$Q_{\nu_\mu}(t) \equiv \frac{1}{2}Q - Q_{f,3}(t)$$

where $Q_{\nu_e}(t) + Q_{\nu_\mu}(t) = Q$.

– We have:

$$Q_{\nu_e} = \cos^2 \theta Q_{\nu_1} + \sin^2 \theta Q_{\nu_2} + \sin \theta \cos \theta \int d^3x \left[ \nu_1^{\dagger} \nu_2 + \nu_2^{\dagger} \nu_1 \right]$$

$$Q_{\nu_\mu} = \sin^2 \theta Q_{\nu_1} + \cos^2 \theta Q_{\nu_2} - \sin \theta \cos \theta \int d^3x \left[ \nu_1^{\dagger} \nu_2 + \nu_2^{\dagger} \nu_1 \right]$$
Hilbert space for mixed neutrinos

- Mixing relations can be written as

$$\nu_e^\alpha(x) = G_\theta^{-1}(t) \nu_1^\alpha(x) G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t) \nu_2^\alpha(x) G_\theta(t)$$

- Mixing generator:

$$G_\theta(t) = \exp[\theta \left(S_+(t) - S_-(t)\right)]$$

- $su(2)$ algebra:

$$S_+(t) \equiv \int d^3x \nu_1^\dagger(x) \nu_2(x) \ , \ S_-(t) \equiv \int d^3x \nu_2^\dagger(x) \nu_1(x)$$

$$S_3 \equiv \frac{1}{2} \int d^3x \left( \nu_1^\dagger(x) \nu_1(x) - \nu_2^\dagger(x) \nu_2(x) \right)$$

$$[S_+(t), S_-(t)] = 2S_3 \ , \ [S_3, S_\pm(t)] = \pm S_\pm(t)$$

\(- \nu_i \) are free Dirac field operators:

\[
\nu_i(x) = \sum_{k,r} \frac{e^{ik \cdot x}}{\sqrt{V}} [\alpha_{k,i}^r(t) + v_{-k,i}^r(t) \beta_{-k,i}^{r\dagger}], \quad i = 1, 2.
\]

- Anticommutation relations:

\[
\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(x - y) \delta_{\alpha\beta} \delta_{ij} \quad ; \quad \{\alpha_{k,i}^r, \alpha_{q,j}^{s\dagger}\} = \{\beta_{k,i}^r, \beta_{q,j}^{s\dagger}\} = \delta^3(k - q) \delta_{rs} \delta_{ij}
\]

- Orthonormality and completeness relations:

\[
\begin{align*}
    u_{k,i}^r(t) &= e^{-i\omega_{k,i} t} u_{k,i}^r \quad ; \quad v_{k,i}^r(t) = e^{i\omega_{k,i} t} v_{k,i}^r \quad ; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2} \\
    u_{k,i}^{r\dagger} u_{k,i}^s &= \delta_{rs} \quad , \quad u_{k,i}^{r\dagger} v_{-k,i}^s = 0 \quad , \quad \sum_r (u_{k,i}^{r\alpha*} u_{k,i}^{r\beta} + v_{-k,i}^{r\alpha*} v_{-k,i}^{r\beta}) = \delta_{\alpha\beta}.
\end{align*}
\]

- Fock space for \( \nu_1, \nu_2 \):

\[
\mathcal{H}_{1,2} = \{ \alpha_{1,2}^{\dagger}, \beta_{1,2}^{\dagger}, |0\rangle_{1,2} \}.
\]
• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G^{-1}_\theta(t) |0\rangle_{1,2} = e^{-\theta(S_+(t)-S_-(t))} |0\rangle_{1,2}$$

The “flavor vacuum” $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state*.

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V \to \infty} \langle 0 | 0(t) \rangle_{e,\mu} = \lim_{V \to \infty} e^{V} \int \frac{d^3 k}{(2\pi)^3} \ln \left(1 - \sin^2 \theta |V_k|^2 \right)^2 = 0$$

with

$$|V_k|^2 \equiv \sum_{r,s} |v_{-k,1}^r u_{k,2}^s|^2 = \frac{k^2 [(\omega_{k,2} + m_2) - (\omega_{k,1} + m_1)]^2}{4 \omega_{k,1} \omega_{k,2} (\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}; \quad 0 \leq |V_k|^2 \leq \frac{1}{2}$$

Quantum Field Theory vs. Quantum Mechanics

- **Quantum Mechanics:**
  - finite \# of degrees of freedom.
  
  - unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- **Quantum Field Theory:**
  - infinite \# of degrees of freedom.
  
  - \( \infty \) many unitarily inequivalent representations of the field algebra \( \Leftrightarrow \) many vacua.
  
  - The mapping between interacting and free fields is “weak”, i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

• Condensate structure of $|0\rangle_{e,\mu}$ (use $\epsilon^r = (-1)^r$):

$$|0\rangle_{e,\mu} = \prod_{k,r} \left[ (1 - \sin^2 \theta |V_k|^2) - \epsilon^r \sin \theta \cos \theta |V_k| (\alpha_{k,1}^r \beta_{-k,2}^r + \alpha_{k,2}^r \beta_{-k,1}^r) ight.$$ 

$$+ \epsilon^r \sin^2 \theta |V_k||U_k| (\alpha_{k,1}^r \beta_{-k,1}^r - \alpha_{k,2}^r \beta_{-k,2}^r) + \sin^2 \theta |V_k|^2 \alpha_{k,1}^r \beta_{-k,2}^r \alpha_{k,2}^r \beta_{-k,1}^r \right] |0\rangle_{1,2}$$

– Orthogonality also for flavor vacua at different times:

$$\lim_{V \to \infty} e,\mu \langle 0(t)|0(t')\rangle_{e,\mu} = 0 \quad \text{for} \quad t \neq t'$$

– Condensation density:

$$e,\mu \langle 0(t)|\alpha_{k,i}^r \alpha_{k,i}^r|0(t)\rangle_{e,\mu} = e,\mu \langle 0(t)|\beta_{k,i}^r \beta_{k,i}^r|0(t)\rangle_{e,\mu} = \sin^2 \theta |V_k|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).
• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{k,e}(t) = \cos \theta \alpha^r_{k,1} + \sin \theta \left( U^*_k(t) \alpha^r_{k,2} + \epsilon^r V_k(t) \beta^{\dagger r}_{-k,2} \right)$$

$$\alpha^r_{k,\mu}(t) = \cos \theta \alpha^r_{k,2} - \sin \theta \left( U_k(t) \alpha^r_{k,1} - \epsilon^r V_k(t) \beta^{\dagger r}_{-k,1} \right)$$

$$\beta^{r}_{-k,e}(t) = \cos \theta \beta^{r}_{-k,1} + \sin \theta \left( U^*_k(t) \beta^{r}_{-k,2} - \epsilon^r V_k(t) \alpha^{r\dagger}_{k,2} \right)$$

$$\beta^{r}_{-k,\mu}(t) = \cos \theta \beta^{r}_{-k,2} - \sin \theta \left( U_k(t) \beta^{r}_{-k,1} + \epsilon^r V_k(t) \alpha^{r\dagger}_{k,1} \right)$$

• Mixing transformation = Rotation + Bogoliubov transformation.

– Bogoliubov coefficients:

$$U_k(t) = u^{r\dagger}_{k,2} u^r_{k,1} e^{i(\omega_k,2-\omega_k,1)t} ; \quad V_k(t) = \epsilon^r u^{r\dagger}_{k,1} v^r_{-k,2} e^{i(\omega_k,2+\omega_k,1)t}$$

$$|U_k|^2 + |V_k|^2 = 1$$
The flavor charge operators are diagonal in the flavor ladder operators:

\[ :Q_\nu_\sigma(t): \equiv \int d^3x \, :\nu_\sigma^\dagger(x) \nu_\sigma(x): \]

\[ = \sum_r \int d^3k \left( \alpha_{k,\sigma}^r(t) \alpha_{k,\sigma}^r(t) - \beta_{-k,\sigma}^r(t) \beta_{-k,\sigma}^r(t) \right), \quad \sigma = e, \mu. \]

Here \( :...:\) denotes normal ordering with respect to the flavor vacuum:

\[ :A:\equiv A - e,\mu \langle 0 | A | 0 \rangle_{e,\mu} \]

Define flavor neutrino states with definite momentum and helicity:

\[ |\nu_{k,\sigma}^r\rangle \equiv \alpha_{k,\sigma}^r(0) |0\rangle_{e,\mu} \]

- Such states are eigenstates of the flavor charges (at \( t=0 \)):

\[ :Q_\nu_\sigma : |\nu_{k,\sigma}^r\rangle = |\nu_{k,\sigma}^r\rangle \]
– We have, for an electron neutrino state:

\[
Q_{k,\nu_\sigma}(t) \equiv \langle \nu_{k,e}^r | :: Q_{\nu_\sigma}(t) :: | \nu_{k,e}^r \rangle \\
= \left| \{ \alpha_{k,\sigma}^r(t), \alpha_{k,e}^{r\dagger}(0) \} \right|^2 + \left| \{ \beta_{-k,\sigma}^{r\dagger}(t), \alpha_{k,e}^{r\dagger}(0) \} \right|^2
\]

● Neutrino oscillation formula (exact result)*:

\[
Q_{k,\nu_e}(t) = 1 - |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right)
\]

\[
Q_{k,\nu_\mu}(t) = |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right)
\]

- For \( k \gg \sqrt{m_1 m_2} \), \( |U_k|^2 \rightarrow 1 \) and \( |V_k|^2 \rightarrow 0 \).

* M.Blasone, in Proc. of the 36th Int. School of Subnuclear Physics, Erice (1998)
Entanglement for flavor states in QFT

– Entanglement for flavor neutrino states in QFT can be expressed by means of the variances of the neutrino charges:* $Q_{\nu_i}, Q_{\nu_\sigma}(t)$

– Variance of $Q_{\nu_i} \rightarrow$ static entanglement:

$$\Delta Q_{\nu_i}(\nu_e) = \langle \nu_{k,e}^r | Q_{\nu_i}^2(t) | \nu_{k,e}^r \rangle - \langle \nu_{k,e}^r | Q_{\nu_i} | \nu_{k,e}^r \rangle^2$$

$$= \cos^2 \theta \sin^2 \theta$$

– Variance of $Q_{\nu_\sigma} \rightarrow$ flavor entanglement:

$$\Delta Q_{\nu_\sigma}(\nu_e)(t) = \langle \nu_{k,e}^r | Q_{\nu_\sigma}^2(t) | \nu_{k,e}^r \rangle - \langle \nu_{k,e}^r | Q_{\nu_\sigma} | \nu_{k,e}^r \rangle^2$$

$$= Q_{\nu_e \rightarrow \nu_e}^k(t) Q_{\nu_e \rightarrow \nu_\mu}^k(t)$$

in formal agreement with results obtained in QM.

*M.Blasone, F. Dell’Anno and S.De Siena, work in progress.
QM vs. QFT flavor entanglement for $|\nu_e(t)\rangle$. 

$\Delta Q_{\nu_e}(\nu_e)(t)$
Conclusions

- Elementary particles are produced as entangled states in the SM;

- Quantification of multipartite entanglement for neutrinos and quarks;

- Neutrino oscillations as a resource for quantum information;

- Extension to QFT using dynamical symmetry approach to entanglement;

- Entanglement vs. inequivalent representations.
Condensation density for mixed fermions

- $V_k = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2-m_1)^2}{m_1 m_2} \to \infty$.
- $|V_k|^2 \simeq \frac{(m_2-m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.