On neutrino mixing, Lorentz invariance and entanglement

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Abstract. On the basis of recent results on neutrino mixing in Quantum Field Theory, we discuss some consequences of the definition of the states for flavor neutrinos as eigenstates of the flavor charges.

1. Introduction
Neutrino mixing and oscillations are nowadays established experimentally [1], although we do not know yet the exact values of parameters as masses and mixing angles [2]. On the other hand, from a theoretical point of view, there are still many open questions about the origin of the mixing itself and various aspects of neutrino oscillations [3].

Another issue is about the nature of mixed neutrino states, i.e. the flavor states. Problems in dealing with such states have been pointed out since long time, however only recently [4] a consistent framework has been developed in which flavor states are made up of a condensate of particle-antiparticle pairs of neutrinos with definite masses. The key point in such an approach is that flavor states turn out to be the eigenstates of the flavor charge operators which can be derived from the Lagrangian using symmetry arguments [5]. This is not true of the usual quantum mechanical (Pontecorvo) states [6].

In this paper, we elaborate on some consequences which arise if one takes the viewpoint that neutrinos are described at a fundamental level by the flavor states. In particular, we consider the effects of such an assumption on the Lorentz invariance and study the entanglement properties of the flavor states.

2. Flavor charges and weak interaction
In this Section, following Ref[6], we consider the flavor charges for mixed neutrinos which enter the charged current weak interaction Lagrangian together with their corresponding charged leptons. We discuss the case of mixing among two generations.

The Lagrangian is written as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, where $\mathcal{L}_0$ is the free lepton Lagrangian

$$\mathcal{L}_0 = (\bar{\nu}_e, \bar{\nu}_\mu) \left( i \gamma^\mu \partial_\mu - M_\nu \right) \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \left( e, \mu \right) \left( i \gamma^\mu \partial_\mu - M_l \right) \begin{pmatrix} e \\ \mu \end{pmatrix},$$

including the neutrino non-diagonal mass matrix $M_\nu$ and the charged leptons mass matrix $M_l$:

$$M_\nu = \begin{pmatrix} m_{\nu_e} & m_{\nu_e \mu} \\ m_{\nu_e \mu} & m_{\nu_\mu} \end{pmatrix} ; \quad M_l = \begin{pmatrix} m_e & 0 \\ 0 & m_\mu \end{pmatrix}.$$
\[ \mathcal{L}_{\text{int}} \text{ is the charged current interaction Lagrangian:} \]

\[ \mathcal{L}_{\text{int}} = \frac{g}{2\sqrt{2}} \left[ W_\rho^+(x) \sigma_{\mu \nu}(1 - \gamma^5) e(x) + W_\mu^+(x) \sigma_{\mu \nu}(1 - \gamma^5) \nu(x) + h.c. \right]. \] (3)

Now, \( \mathcal{L} \) is invariant under the global phase transformations:

\[ e(x) \rightarrow e^{i\alpha} e(x), \quad \nu_e(x) \rightarrow e^{i\alpha} \nu_e(x), \] (4)

together with

\[ \mu(x) \rightarrow e^{i\alpha} \mu(x), \quad \nu_\mu(x) \rightarrow e^{i\alpha} \nu_\mu(x). \] (5)

These are generated by

\[ Q_e(t) = \int d^4x \, e^\dagger(x) e(x), \quad Q_{\nu_e}(t) = \int d^4x \, \nu_e^\dagger(x) \nu_e(x), \]
\[ Q_\mu(t) = \int d^4x \, \mu^\dagger(x) \mu(x), \quad Q_{\nu_\mu}(t) = \int d^4x \, \nu_\mu^\dagger(x) \nu_\mu(x), \] (6)

(7)

respectively. The invariance of the Lagrangian is then expressed by \([Q^\text{tot}_i, \mathcal{L}(x)] = 0\), which guarantees the conservation of total lepton number. \( Q^\text{tot}_i \) is the total flavor charge:

\[ Q^\text{tot}_i = Q_e(t) + Q_{\nu_e}(t) + Q_\mu(t) + Q_{\nu_\mu}(t) = Q^\text{tot}_e(t) + Q^\text{tot}_\mu(t), \] (8)

Note that the form of the flavor charges (6), (7) is the same as in the case where the mixing is absent [6] and that the presence of the mixed neutrino mass term, i.e. of the non-diagonal mass matrix \( M_\nu \), prevents the invariance of the Lagrangian \( \mathcal{L}_0 \) under the separate phase transformations (4) and (5). Indeed we have:

\[ [Q^\text{tot}_e(t), \mathcal{L}_0(x)] \neq 0, \quad [Q^\text{tot}_\mu(t), \mathcal{L}_0(x)] \neq 0. \] (9)

However, the charges \( Q^\text{tot}_e \) and \( Q^\text{tot}_\mu \), even in the presence of the mixing in the neutrino sector, still commute separately with the interaction Lagrangian \( \mathcal{L}_{\text{int}} \) (at equal times):

\[ [Q^\text{tot}_e(t), \mathcal{L}_{\text{int}}(x)] = 0, \quad [Q^\text{tot}_\mu(t), \mathcal{L}_{\text{int}}(x)] = 0. \] (10)

This implies that, even in presence of mixing, a flavor neutrino state can be well defined in the production vertex as an eigenstate of the neutrino flavor charge \( Q_\nu_e \) for electron neutrinos, \( Q_\nu_\mu \) for muon neutrinos. In practice, such a situation is realized when, as usually it happens, the spatial extension of the neutrino source is much smaller than the neutrino oscillation length.

3. Flavor states for mixed neutrinos

We now turn to the definition of the neutrino flavor states as eigenstates of the flavor charges \( Q_{\nu_e} \) and \( Q_{\nu_\mu} \). To this aim, it is sufficient to study the neutrino part of the free Lagrangian Eq. (1). When mixing is turned off, we have two free neutrino fields \( \nu_1, \nu_2 \) with masses \( m_{\nu_1}, m_{\nu_2} \). The normal ordered charge operators for such fields are:

\[ Q_{\nu_i} := \int d^4x : \nu_i^\dagger(x) \nu_i(x) := \sum_r \int d^4k \left( \alpha^r_{+,k,i} \alpha^r_{+,k,i} - \beta^r_{-,k,i} \beta^r_{-,k,i} \right), \] (11)
where \( i = 1, 2 \) and \( : \ldots : \) denotes normal ordering with respect to the vacuum \(|0\rangle_{1,2}\) for the free fields \( \nu_1, \nu_2 \). The neutrino states with definite masses defined as \( |\nu_{r,i}^{\nu_1}\rangle \equiv \alpha_{r,i}^{\nu_1}|0\rangle_{1,2}, i = 1, 2 \), are then eigenstates of \( Q_{\nu_1} \) and \( Q_{\nu_2} \), which are the (conserved) lepton charges of neutrinos in the absence of mixing.

The situation is more delicate when mixing is present. The flavor charges in the presence of mixing (6), (7) can be related to \( Q_{\nu_1} \) and \( Q_{\nu_2} \) as [5]

\[
Q_{\nu_e}(t) = \cos^2 \theta \, Q_{\nu_1} + \sin^2 \theta \, Q_{\nu_2} + \sin \theta \cos \theta \int d^3x \left[ \nu_1^\dagger(x) \nu_2(x) + \nu_2^\dagger(x) \nu_1(x) \right],
\]

\[
Q_{\nu_\mu}(t) = \sin^2 \theta \, Q_{\nu_1} + \cos^2 \theta \, Q_{\nu_2} - \sin \theta \cos \theta \int d^3x \left[ \nu_1^\dagger(x) \nu_2(x) + \nu_2^\dagger(x) \nu_1(x) \right].
\]

The last term in these expressions forbids the construction of eigenstates of the \( Q_{\nu_e}(t) \) in the Hilbert space \( \mathcal{H}_{1,2} \). This can be understood by considering that the mixing transformations

\[
\nu_e(x) = \cos \theta \, \nu_1(x) + \sin \theta \, \nu_2(x)
\]

\[
\nu_\mu(x) = -\sin \theta \, \nu_1(x) + \cos \theta \, \nu_2(x)
\]

can be also implemented in the following way [4]::

\[
\nu_\sigma(x) \equiv G_{\theta}^{-1}(t) \nu_j(x) G_{\theta}(t), \quad (\sigma, j) = (e, 1), (\mu, 2)
\]

\[
G_{\theta}(t) = \exp \left[ \theta \int d^3x \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right].
\]

with \( t \equiv x^0 \). The mixing generator \( G_{\theta}(t) \) allows to define flavor ladder operators and the flavor vacuum as [4]:

\[
\alpha_{k,\sigma}^r(t) \equiv G_{\theta}^{-1}(t) \alpha_{k,j}^r(t) G_{\theta}(t) ; \quad \beta_{-k,\sigma}^r(t) \equiv G_{\theta}^{-1}(t) \beta_{-k,j}^r(t) G_{\theta}(t),
\]

\[
|0(t)\rangle_{e,\mu} \equiv G_{\theta}^{-1}(t) |0\rangle_{1,2}, \quad (\sigma, j) = (e, 1), (\mu, 2)
\]

The flavor vacuum \(|0\rangle_{e,\mu} \) is orthogonal to the mass vacuum \(|0\rangle_{1,2}\) in the infinite volume limit [4].

The normal ordered flavor charge operators for mixed neutrinos are then written as

\[
z \cdot Q_{\nu_\sigma}(t) = \sum_r \int d^3k \left( \alpha_{k,\sigma}^r(t) \alpha_{-k,\sigma}^r(t) - \beta_{-k,\sigma}^r(t) \beta_{k,\sigma}^r(t) \right)
\]

where \( \sigma = e, \mu \), and \( : \ldots : \) denotes normal ordering with respect to \(|0\rangle_{e,\mu} \). Thus, the flavor charges are diagonal in the flavor annihilation/creation operators above introduced.

Note that \( z \cdot Q_{\nu_\sigma}(t) = G_{\theta}^{-1}(t) \circ \circ \cdot G_{\theta}(t) \), with \( (\sigma, j) = (e, 1), (\mu, 2) \), and

\[
z \cdot Q_\nu = z \cdot Q_{\nu_\sigma}(t) = + \cdot Q_{\nu_{\mu}}(t) = + : Q_{\nu_{\mu}} : = : Q_{\nu} :.
\]

The flavor states are defined as eigenstates of the flavor charges \( Q_{\nu_\sigma} \) at a reference time \( t = 0 \):

\[
|\nu_{k,\sigma}^r\rangle \equiv \alpha_{k,\sigma}^r(t)|0\rangle_{e,\mu}, \quad \sigma = e, \mu
\]

and similar ones for antiparticles. The explicit form of \( |\nu_{k,e}^r\rangle \) and \( |\nu_{k,\mu}^r\rangle \) can be found in Ref.[6].
4. Lorentz invariance for mixed neutrinos

The above defined flavor states turn out to be eigenstates of the momentum operator \( \hat{p} \). However, their energies are not sharply defined since they represent mixed states. Note indeed that we have \( [Q_{\nu_e}(t), \hat{H}] \neq 0 \), where \( \hat{H} \) is the neutrino free Hamiltonian. Therefore, one has two alternatives: to work either with mass eigenstates or with (lepton) charge eigenstates. We have chosen in the above this second possibility, motivated by the form of the charged current weak interaction (which is diagonal in the flavor).

So, if the neutrino flavor states are to be taken as fundamental, one may wonder that some form of violation of Lorentz invariance will arise for such non-conventional states. Indeed, they have no definite energy, although mean values can be defined as:

\[
E_{\nu_e}(k) \equiv \langle \nu_{\nu_e}^* | \hat{H} | \nu_{\nu_e} \rangle = \omega_{k,1} \cos^2 \theta + (1 - 2|V_{k}|^2) \omega_{k,2} \sin^2 \theta ,
\]

\[
E_{\nu_\mu}(k) \equiv \langle \nu_{\nu_\mu}^* | \hat{H} | \nu_{\nu_\mu} \rangle = \omega_{k,2} \cos^2 \theta + (1 - 2|V_{k}|^2) \omega_{k,1} \sin^2 \theta .
\]

In Ref.[7, 8] these have been taken as modified dispersion relations and the corresponding non-linear realization of the Lorentz algebra has been studied.

From Eqs. (21), (22), we define the “rest masses” for the mixed neutrinos:

\[
m_{\nu_e} \equiv E_{\nu_e}(k = 0) = m_1 \cos^2 \theta + m_2 \sin^2 \theta ,
\]

\[
m_{\nu_\mu} \equiv E_{\nu_\mu}(k = 0) = m_2 \cos^2 \theta + m_1 \sin^2 \theta .
\]

It has been shown [7, 8] that such masses could be a signature of the fundamental nature of flavor states. Indeed, for example, \( m_{\nu_e} \) is expected to appear as end-point of the electron spectrum in a beta decay, instead of the lightest mass eigenvalue \( m_{\nu_1} \) as would be in the case when the mass eigenstates are taken to be fundamental.

5. Neutrino oscillations and Lorentz invariance

In the above framework, neutrino oscillations are described by the expectation values of the flavor charges at a later time on the neutrino state taken at an initial reference time, as for example \( \langle \nu_{\nu_e}^* Q_{\nu_e}(t) | \nu_{\nu_e} \rangle \).

The flavor charges above defined are related to the corresponding currents as [5]:

\[
Q_{\nu_e}(t) = \int d^3x J_{\nu_e}^\sigma(x) , \quad \sigma = e, \mu
\]

where \( J_{\nu_e}^\sigma(x) \equiv \bar{\nu}_e(x) \gamma^\mu \nu_e(x) = G^{-1}_\theta(t) J_{\nu_e}^\sigma(0) G_\theta(t) \).

Since \( Q_{\nu_e} \) depend explicitly on time, the above expectation value giving the oscillation formula cannot be a Lorentz scalar. We observe that the flavor currents have non-zero divergence:

\[
\partial_\mu J_{\nu_e}^\mu(x) = \partial_\mu G^{-1}_{\nu_e}(t) J_{\nu_e}^\mu(x) G_\theta(t) = [J_{\nu_e}^0(x), G^{-1}_\theta(t) G_\theta(t)]
\]

where we denoted by a dot the time derivative and have used the fact that \( G^{-1} \dot{G} = -\dot{G}^{-1} G \).

The above term can be calculated explicitly:

\[
\partial_\mu J_{\nu_e}^\mu(x) = i(m_2 - m_1) \sin \theta \cos \theta [\bar{\nu}_2(x) \nu_1(x) - \bar{\nu}_1(x) \nu_2(x)] = -\partial_\mu J_{\nu_\mu}^\mu(x) ,
\]

We now consider the situation in which there are two inertial observers, \( O \) and \( O' \), related by a Lorentz transformation and relate the flavor charges defined by both of them.
Let us denote with \( Q_{\nu_e}(t) \) the (electron neutrino) charge defined in a reference frame and by \( Q'_{\nu_e}(t') \) the corresponding quantity defined in the transformed frame. We obtain

\[
Q_{\nu_e}(t) = \int_{\Sigma} J_{\nu_e}^\mu(x) d\Sigma^\mu = \int_{\Sigma'} J'_{\nu_e}^\mu(x') d\Sigma'^\mu + \int_{\Omega} d^4x \partial_p J_{\nu_e}^\mu(x) = Q'_{\nu_e}(t') + \int_{\Omega} d^4x \partial_p J_{\nu_e}^\mu(x),
\] (28)

where \( \Sigma \) and \( \Sigma' \) are two space-like hypersurfaces defined by \( t = \text{const.} \) and \( t' = \text{const.} \), respectively. \( \Omega \) is the volume delimited by them. A similar relation is valid for \( Q_{\nu_e} \).

The above expressions are to be compared with the corresponding relations for the Noether charges \( Q_{\nu_1} \) and \( Q_{\nu_2} \), which represent the flavor charges in absence of mixing: these are time-independent and Lorentz scalars since the corresponding currents are divergenceless.

Note that the (time-independent) total leptonic charge \( Q_e = Q_{\nu_e}(t) + Q'_{\nu_e}(t) \) is a Lorentz scalar and consequently all inertial observers agree on its value when measured on a flavor neutrino state. On the other hand, the flavor charges do not need to have the same value in every inertial frame, as shown by Eq. (28).

One can then calculate the correction to the oscillation formula for a neutrino coming from a moving (boosted) source \( [9] \). The correction term disappears in the extreme relativistic limit \( |k| \gg \sqrt{m_1 m_2} \), thus recovering the result of Ref. [10].

6. Flavor states as entangled states

The assumption of the flavor states as the fundamental entities for neutrinos, leads to an interesting question about the nature of such states as entangled states. A detailed analysis of this aspect is beyond the scope of the present work. Here we only present some elementary considerations on the entanglement properties of the usual quantum mechanical (Pontecorvo) neutrino states:

\[
|\nu_e \rangle = \cos \theta |\nu_1 \rangle + \sin \theta |\nu_2 \rangle \quad (29)
\]

\[
|\nu_\mu \rangle = -\sin \theta |\nu_1 \rangle + \cos \theta |\nu_2 \rangle \, , \quad (30)
\]

These states live in the tensor product Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) and are both bipartite entangled states. In fact they can be written as follows (we omit momentum and spin indices):

\[
|\nu_e \rangle = \cos \theta |1 \rangle_1 \otimes |0 \rangle_2 + \sin \theta |0 \rangle_1 \otimes |1 \rangle_2 \quad (31)
\]

\[
|\nu_\mu \rangle = -\sin \theta |1 \rangle_1 \otimes |0 \rangle_2 + \cos \theta |0 \rangle_1 \otimes |1 \rangle_2 \quad (32)
\]

Next we can consider the density matrices for these pure states:

\[
\rho^e \equiv |\nu_e \rangle \langle \nu_e | \quad ; \quad \rho^\mu \equiv |\nu_\mu \rangle \langle \nu_\mu |
\] (33)

The reduced density matrices can be easily computed with the result

\[
\rho^e = \cos^2 \theta |1 \rangle_1 \langle 1 |_1 + \sin^2 \theta |0 \rangle_1 \langle 0 |_1 \, , \quad \rho^\mu = \sin^2 \theta |1 \rangle_2 \langle 1 |_2 + \cos^2 \theta |0 \rangle_2 \langle 0 |_2 \quad (34)
\]

\[
\rho^e = \sin^2 \theta |1 \rangle_1 \langle 1 |_1 + \cos^2 \theta |0 \rangle_1 \langle 0 |_1 \, , \quad \rho^\mu = \cos^2 \theta |1 \rangle_2 \langle 1 |_2 + \sin^2 \theta |0 \rangle_2 \langle 0 |_2 \quad (35)
\]

where \( \rho^e \equiv Tr_2(\rho^e) = \sum_j j (|\nu_e \rangle \langle \nu_e |) |j \rangle 2 \) etc.

We can now compute the von Neumann entropy \( S(\rho) = -Tr(\rho \log_2 \rho) \) for the flavor states:

\[
S(\rho^e) = S(\rho^\mu) = S(\rho^e) = -\cos^2 \theta \log_2 \cos^2 \theta - \sin^2 \theta \log_2 \sin^2 \theta \quad (36)
\]
Note that for maximal mixing $\theta = \pi/4$, the above flavor states reduce to the states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ of the Bell basis:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$  

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

In such a case the von Neumann entropy takes the maximal value 1 and the entanglement is maximal.

The case of three flavor mixing is more complicated since it represents a tripartite system. We choose to work with the following (CKM) parametrization of the mixing matrix [11]:

$$\left( \begin{array}{c} |\nu_e\rangle \\ |\nu_{\mu}\rangle \\ |\nu_{\tau}\rangle \end{array} \right) = \left( \begin{array}{ccc} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{array} \right) \left( \begin{array}{c} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{array} \right).$$

In this case, we have three mixing angles plus a phase responsible for CP violation. The values of these parameters for which the mixing is maximal are:

$$\theta_{12}^{\text{max}} = \frac{\pi}{4}; \quad \theta_{23}^{\text{max}} = \frac{\pi}{4}; \quad \theta_{13}^{\text{max}} = \arccos \sqrt{\frac{2}{3}}; \quad \delta^{\text{max}} = \frac{\pi}{2}.$$

In correspondence of these values, the CKM matrix elements have all the same modulus $\frac{1}{\sqrt{3}}$:

$$U^{\text{max}} = \frac{1}{\sqrt{3}} \left( \begin{array}{ccc} 1 & 1 & -1 \\ iy & iy^2 & 1 \\ iy^2 & iy & 1 \end{array} \right).$$

with $y = \exp(2i\pi/3)$. Entangled states of the form (39) are classified as $|W\rangle$ states.

It is an interesting question to ask if the maximal mixing case corresponds also in such a case to maximally entangled states. Work is in progress along this direction.

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References