

# Quantum mechanics of the damped harmonic oscillator

Massimo Blasone and Petr Jizba

**Abstract:** We quantize the system of a damped harmonic oscillator coupled to its time-reversed image, known as Bateman's dual system. By using the Feynman–Hibbs method, the time-dependent quantum states of such a system are constructed entirely in the framework of the classical theory. The geometric phase is calculated and found to be proportional to the ground-state energy of the one-dimensional linear harmonic oscillator to which the two-dimensional system reduces under appropriate constraint.

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**Résumé :** Nous quantifions le système composé de l'oscillateur harmonique amorti couplé à son image renversée dans le temps, connu sous le nom de système dual de Bateman. La méthode de Feynman–Hibbs permet la construction complète dans un cadre classique des états quantiques dépendants du temps du système. Nous calculons la phase géométrique et la trouvons proportionnelle à l'énergie du fondamental de l'oscillateur harmonique linéaire 1D auquel ce système se réduit sous un ensemble approprié de contraintes.

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## 1. Introduction

The study of dissipative systems and their quantization is of a great theoretical and practical value in view of the many different situations in which dissipative phenomena with a quantum origin manifest themselves. These include particle decays, phase transitions, decoherence phenomena, plasma physics (both in quantum electrodynamics and in quantum chromodynamics), self-organization, and pattern formation in biological and chemical systems. In spite of this, the difficulty inherent in quantization of dissipative systems is seemingly unavoidable. As a matter of fact, even the quantization of the simplest dissipative system — the damped harmonic oscillator (DHO) — is not an easy task [1], and indeed in the course of time, various approaches have been devised to deal with this problem. When trying to set up a Lagrangian (or Hamiltonian) formalism for the DHO one finds already at a classical level that it is possible to proceed in different directions. For instance, one can double the phase-space dimensions [1–4] to include the environment, or one may use an explicitly time-dependent Hamiltonian [1, 5] to account for irreversibility.

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In the present paper, we follow the first route and consider the system of a one-dimensional damped oscillator coupled to its time-reversed image, known as Bateman's dual system [2]. This system represents a simple explicit example of a dissipative system that can be tackled by means of canonical quantization [1, 3, 4]. In addition, interesting connections with (Chern–Simons) gauge theories emerge [6]. However, our pivotal motivation to study Bateman's model resides in the recent proposal of 't Hooft about deterministic Quantum Mechanics (QM) [7]. In this connection, Bateman's system has revealed a simple (although nontrivial) prototype for implementing 't Hooft's construction [8]: from the (deterministic) system of two dissipative harmonic oscillators one obtains, under an appropriate reduction of the Hilbert space, a (fully quantum) linear harmonic oscillator (LHO).

The aim of the present paper is to analyze the QM structure of Bateman's system by using the Feynman–Hibbs kernel approach, which is based on a direct quantization of (classical) equations of motion. This seems to be a convenient direct way to quantize dissipative systems [9]. Our results include the time-dependent wave function and the exact geometric phases associated with them. We also study the reduction of the two-dimensional Bateman system to the one-dimensional linear harmonic oscillator (LHO).

The outline of the paper is as follows. In Sect. 2, we quantize Bateman's system by using the Feynman–Hibbs prescription for the time-evolution amplitude (kernel) [10–12]. We show that the kernel is fully expressible in terms of solutions of the classical equations of motion and that it is invariant under their choice. This might be viewed as a two-dimensional extension of an existing one-dimensional result [12].

In Sect. 3, we obtain the time-dependent wave functions in hyperbolic radial coordinates  $(r, u)$ . They are expressed in terms of generalized Laguerre polynomials and are shown to satisfy the "correct" time-dependent Schrödinger equation. We also show that a new inner product has to be defined to deal with the dissipative nature of the wave functions. In fact, one of the merits of the method presented is that it naturally provides such a inner product. In Sect. 4, we discuss other features of Bateman's system, including the behavior under time-reversal and connection with  $SU(1, 1)$  squeezed states.

In Sect. 5, we calculate the (Pancharatnam) geometric phase for Bateman's dual system. Finally, in Sect. 6, we study the reduction to the one-dimensional LHO and show that the corresponding (Berry) geometric phase bears an imprint of the original two-dimensional system even after the performed reduction to the one-dimensional LHO. The geometric phase thus obtained can be directly identified with the zero point energy of the one-dimensional LHO, and, in general, is different from the usual  $E_0 = \hbar\Omega/2$ . This is in line with the analysis in ref. 8.

## 2. Kernel for Bateman's dual system

We start by recalling some features of the Feynman–Hibbs quantization for systems described by Hamiltonians, which can possibly be time-dependent or even non-Hermitian. Let us consider the *time-evolution amplitudes* (or simply *kernels*)

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle \equiv \langle \mathbf{x}_b | U(t_b, t_a) | \mathbf{x}_a \rangle \quad (1)$$

Since the time-evolution operator  $U(t_b, t_a)$  fulfills the Schrödinger equations

$$i\hbar \frac{\partial}{\partial t_b} U(t_b, t_a) = \hat{H} U(t_b, t_a) \quad \text{and} \quad i\hbar \frac{\partial}{\partial t_b} U(t_a, t_b) = -U(t_a, t_b) \hat{H}, \quad t_b > t_a \quad (2)$$

the kernel satisfies [13]

$$i\hbar \frac{\partial}{\partial t_b} \langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \hat{H} (-i\hbar \partial_{\mathbf{x}_b}, \mathbf{x}_b) \langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle \quad (3)$$

$$i\hbar \frac{\partial}{\partial t_b} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle = -\mathcal{T} \hat{H}^\dagger (-i\hbar \partial_{\mathbf{x}_b}, \mathbf{x}_b) \mathcal{T}^{-1} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle, \quad t_b > t_a \quad (4)$$

with the initial condition  $\lim_{t_b \rightarrow t_a} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle = \delta(\mathbf{x}_a - \mathbf{x}_b)$ . Here  $\mathcal{T}$  is the (anti-unitary) time reversal operator and  $\hat{H}^\dagger$  is the Hermitian-adjoint Hamiltonian. For quadratic Hamiltonians the kernel has a very simple form

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = F[t_a, t_b] \exp\left(\frac{i}{\hbar} S_{\text{cl}}[\mathbf{x}]\right) \quad t_b > t_a \quad (5)$$

The function  $F[t_a, t_b]$  is the so called *fluctuation factor* [11] and is independent of both  $\mathbf{x}_a$  and  $\mathbf{x}_b$  [10, 11]. For a system with a time-independent Hamiltonian, the kernel reads

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \langle \mathbf{x}_b | \exp\left(-\frac{i}{\hbar} \hat{H}(t_b - t_a)\right) | \mathbf{x}_a \rangle \quad (6)$$

Inserting the resolution of unity  $\sum_m |\psi_m\rangle \langle \psi_m| = 1$ , ( $|\psi_m\rangle$  are orthonormal base kets at  $t = 0$  spanning the Hilbert space) into (6), we find that

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \sum_m \psi_m(\mathbf{x}_b, t_b) \psi_m^*(\mathbf{x}_a, t_a) \quad (7)$$

with an identification  $\psi_m(\mathbf{x}, t) = \langle \mathbf{x} | \psi_m(t) \rangle$ . The symbol (\*) does not necessarily correspond to the complex conjugation. This is because  $\psi_m(\mathbf{x}, t)$  and  $\psi_m^*(\mathbf{x}, t)$  obey (3) and (4), respectively, and in dissipative systems these are not generally complex conjugates of each other.

We now apply the Feynman–Hibbs prescription (5) to the quantization of the Bateman dual system, which describes a two-dimensional interacting system of damped-amplified harmonic oscillators. The corresponding Lagrangian reads [1–4, 6, 14]

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - \kappa xy \quad (8)$$

giving the (classical) equations of motion

$$m\ddot{x}_{\text{cl}} + \gamma\dot{x}_{\text{cl}} + \kappa x_{\text{cl}} = 0 \quad (9)$$

$$m\ddot{y}_{\text{cl}} - \gamma\dot{y}_{\text{cl}} + \kappa y_{\text{cl}} = 0 \quad (10)$$

Note that with appropriate initial conditions the  $y$  system corresponds to a time-reversed image of the damped oscillator  $x$ . One may think of  $y$  as describing an effective degree of freedom for the reservoir to which the system with the  $x$  degree of freedom is coupled [1, 4, 6]. For future convenience, we introduce the reduced oscillators frequency  $\Omega = \sqrt{\frac{\kappa}{m} - \frac{\gamma^2}{4m^2}}$  and the damping factor  $\Gamma = \frac{\gamma}{2m}$ . We will assume  $\Omega$  to be real (under-damped case).

It is useful to work with the rotated variables [6]  $x_1 = (x + y)/\sqrt{2}$ ,  $x_2 = (x - y)/\sqrt{2}$ . Then

$$L = \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \frac{\gamma}{2}(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \frac{\kappa}{2}(x_1^2 - x_2^2) = \frac{m}{2} \dot{\mathbf{x}} \dot{\mathbf{x}} + \frac{\gamma}{2} \mathbf{x} \wedge \dot{\mathbf{x}} - \frac{\kappa}{2} \mathbf{x} \mathbf{x} \quad (11)$$

where we have introduced the notation  $\mathbf{a} \mathbf{b} = g_{\alpha\beta} a^\alpha b^\beta$ ,  $\mathbf{a} \wedge \mathbf{b} = \varepsilon^{\alpha\beta} a_\alpha b_\beta$ , and  $x^\alpha = (x_1, x_2)$  with the metric tensor  $g_{\alpha\beta} = (\sigma_3)_{\alpha\beta}$  (note also that  $\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta}$ ). In  $(x_1, x_2)$  coordinates, the conjugate momenta, equations of motion and the classical action are

$$\mathbf{p} = m\dot{\mathbf{x}} - \frac{1}{2}\gamma\sigma_1 \mathbf{x} \quad (12)$$

$$m \ddot{\mathbf{x}}_{\text{cl}} + \gamma \sigma_1 \dot{\mathbf{x}}_{\text{cl}} + \kappa \mathbf{x}_{\text{cl}} = \mathbf{0} \quad (13)$$

$$S_{\text{cl}}[\mathbf{x}] = \int_{t_a}^{t_b} dt L = \frac{m}{2} [\mathbf{x}_{\text{cl}}(t_b) \dot{\mathbf{x}}_{\text{cl}}(t_b) - \mathbf{x}_{\text{cl}}(t_a) \dot{\mathbf{x}}_{\text{cl}}(t_a)] \quad (14)$$

Following the procedure used in ref. 12, we now build the kernel in terms of a fundamental system of solutions (i.e., a maximal system of linearly independent solutions) of (13). The latter consists of four real  $1 \times 2$  vectors  $\mathbf{u}_i, \mathbf{v}_i$ , with ( $i = 1, 2$ ). The independence of the solutions is checked via the Wronskian, which in our case is the determinant of the  $4 \times 4$  matrix

$$W(t) = W(t_0) = \begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{v}_1 & \mathbf{v}_2 \\ \dot{\mathbf{u}}_1 & \dot{\mathbf{u}}_2 & \dot{\mathbf{v}}_1 & \dot{\mathbf{v}}_2 \end{vmatrix} \quad (15)$$

which has to be nonzero (in fact for (13) the Wronskian is time-independent [15]). In ref. 15 it is shown that any real solution  $\mathbf{x}_{\text{cl}}(t)$  of (13) with two fixed points  $\mathbf{x}_{\text{cl}}(t_a) \equiv \mathbf{x}_a$  and  $\mathbf{x}_{\text{cl}}(t_b) \equiv \mathbf{x}_b$  might be written as

$$\mathbf{x}_{\text{cl}}(t) = \frac{[x_a^1 \mathbf{B}_1(t) + x_a^2 \mathbf{B}_2(t) + x_b^1 \mathbf{B}_3(t) + x_b^2 \mathbf{B}_4(t)]}{U_a V_b} \quad (16)$$

where  $U_a = |\mathbf{u}_1(t_a) \mathbf{u}_2(t_a)|$ ,  $V_b = |\mathbf{v}_1(t_b) \mathbf{v}_2(t_b)|$ , and  $\mathbf{B}_i(t) = \begin{pmatrix} B_i^1(t) \\ B_i^2(t) \end{pmatrix}$ . The  $B_i^1$  and  $B_i^2$  are given by the determinant  $D$

$$D = \begin{vmatrix} \mathbf{u}_1(t_a) & \mathbf{u}_2(t_a) & \mathbf{0} & \mathbf{0} \\ \mathbf{u}_2(t_b) & \mathbf{u}_2(t_b) & \mathbf{v}_1(t_b) & \mathbf{v}_2(t_b) \end{vmatrix} = U_a V_b \quad (17)$$

with the  $i$ th row substituted by  $(u_1^1(t), u_2^1(t), v_1^1(t), v_2^1(t))$  or  $(u_1^2(t), u_2^2(t), v_1^2(t), v_2^2(t))$ , respectively. As a result, the classical action  $S_{\text{cl}}[\mathbf{x}]$  might be written as

$$S_{\text{cl}}[\mathbf{x}] = \frac{m}{2D} \left[ x_a^1 \mathbf{x}_b \dot{\mathbf{B}}_1(t_b) + x_a^2 \mathbf{x}_b \dot{\mathbf{B}}_2(t_b) + x_b^1 \mathbf{x}_b \dot{\mathbf{B}}_3(t_b) + x_b^2 \mathbf{x}_b \dot{\mathbf{B}}_4(t_b) - x_a^1 \mathbf{x}_a \dot{\mathbf{B}}_1(t_a) - x_a^2 \mathbf{x}_a \dot{\mathbf{B}}_2(t_a) - x_b^1 \mathbf{x}_a \dot{\mathbf{B}}_3(t_a) - x_b^2 \mathbf{x}_a \dot{\mathbf{B}}_4(t_a) \right] \quad (18)$$

Using the basic properties of determinants it is possible to show [15] that both  $S_{\text{cl}}[\mathbf{x}]$  and  $\mathbf{x}_{\text{cl}}(t)$  are independent of the choice of the fundamental system of solutions.

Now, to determine the kernel, we still need to calculate the fluctuation factor  $F[t_a, t_b]$ . This can be done by employing the Van Vleck–Pauli–Morette determinant [16–18]

$$F[t_a, t_b] = \sqrt{\det_2 \left( \frac{i}{2\pi \hbar} \frac{\partial^2 S_{\text{cl}}}{\partial \mathbf{x}_a^\alpha \partial \mathbf{x}_b^\beta} \right)} \quad (19)$$

The symbol  $\det_2(\dots)$  denotes here a  $2 \times 2$  determinant. It is important to note that (19) is correct only for small  $t_b - t_a$  [19]. In the general case, the determinant on the right-hand side of (19) will become infinite every time the classical (position space) orbit touches (or crosses) a caustic.<sup>3</sup> Scrutiny of the quadratic systems performed in refs. 11 and 20 revealed that (5) remains valid even after passing through a caustic, provided one writes the fluctuation factor as

$$F[t_a, t_b] = \sqrt{\left| \det_2 \left( \frac{i}{2\pi \hbar} \frac{\partial^2 S_{\text{cl}}}{\partial \mathbf{x}_a^\alpha \partial \mathbf{x}_b^\beta} \right) \right|} \quad (20)$$

<sup>3</sup>The set of all points where the inverse of the Van Vleck–Pauli–Morette determinant vanishes is called a caustic. The Morse index then counts how many times the classical orbit crosses (or touches) the caustic when passing from the initial to the final position. In the literature, crossing points are often called focal or conjugate points.

and insert a factor  $\exp(-i\pi/2)$  for every reduction of the rank of  $1/\det_2(\partial^2 S_{\text{cl}}/\partial \mathbf{x}_a^\alpha \partial \mathbf{x}_b^\beta)$  at the caustic

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \exp -i(\pi/2)n_{a,b} F[t_a, t_b] \exp\left(\frac{i}{\hbar} S_{\text{cl}}[\mathbf{x}]\right), \quad t_b > t_a \quad (21)$$

Here  $n_{a,b}$  is the Morse (or Maslov) index [11, 20–24] of the classical path running from  $\mathbf{x}_a$  to  $\mathbf{x}_b$ . To simplify the discussion, we omit, for a while, the delicate issue of caustics assuming that the determinant in (19) is positive. In such a case the explicit calculation of the fluctuation factor gives [15]

$$F[t_a, t_b] = \frac{m}{2\pi\hbar} \sqrt{\frac{W}{D}} \quad (22)$$

We shall, however, return to the caustic issue in Sects. 5 and 6.

### 3. Time-dependent wave functions of Bateman's dual system

To compute wave functions, it is useful to rewrite the kernel in hyperbolic polar coordinates  $(r, u)$ , where  $x_1 = r \cosh u$  and  $x_2 = r \sinh u$ , and then implement the defining relation [10]

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \sum_{n,l} \psi_{n,l}(r_b, u_b, t_b) \psi_{n,l}^{(*)}(r_a, u_a, t_a), \quad t_b > t_a \quad (23)$$

For reasons, which will become clear shortly, we have used the symbol  $(*)$  instead of the usual complex conjugation  $*$ . From (5), (18), and (22) we obtain [15]

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \frac{m}{2\pi\hbar} \sqrt{\frac{W}{D}} \exp\left[\frac{i m}{2\hbar} \left(-[r_a^2 + r_b^2] \frac{\dot{B}_1^1(t_a)}{D} + 2r_a r_b \left[\frac{\dot{B}_1^1(t_b)}{D} \cosh(\Delta u) - \frac{\dot{B}_1^2(t_b)}{D} \sinh(\Delta u)\right]\right)\right] \quad (24)$$

with  $\Delta u = u_b - u_a$  and  $t_b > t_a$ . By observing that  $[\dot{B}_1^1(t_b)]^2 - [\dot{B}_1^2(t_b)]^2 = WD$  we can write  $\dot{B}_1^1(t_b) \equiv \sqrt{WD} \cosh \alpha$  and  $\dot{B}_1^2(t_b) \equiv \sqrt{WD} \sinh \alpha$ . One can then show [15] that  $\alpha(t_a, t_b) = \Gamma(t_a - t_b) + \beta$  with  $\Gamma = \frac{\gamma}{2m}$  and  $\beta$  a complex (time-independent) constant. Inasmuch, we may rewrite the kernel (24) in the following form:

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \frac{m}{2\pi\hbar} \sqrt{\frac{W}{D}} \exp\left[-\frac{i m}{4D\hbar} \left(\frac{dD}{dt_a} r_a^2 - \frac{dD}{dt_b} r_b^2\right) + \frac{i m}{\hbar} \sqrt{\frac{W}{D}} r_a r_b \cosh(\Delta u - \alpha)\right] \quad (25)$$

Further simplification might be reached if we employ the Laurent expansion [14, 25]<sup>4</sup>

$$\exp(i a \cosh(u)) = \sum_{l=-\infty}^{\infty} (-1)^l I_l(-i a) e^{-lu} \quad (26)$$

where  $I_l(\dots)$  are the modified Bessel functions. We then use the addition theorem for the generalized Laguerre polynomials  $L_n^l$  (see, for example, refs. 14 and 25)

$$\sum_{n=0}^{\infty} n! \frac{L_n^l(z_1) L_n^l(z_2) b^n}{\Gamma(n+l+1)} = \frac{(z_1 z_2 b)^{-l/2}}{1-b} \exp\left[-b \frac{z_1 + z_2}{1-b}\right] I_l\left(2 \sqrt{\frac{z_1 z_2 b}{1-b}}\right) \quad (27)$$

<sup>4</sup>Because  $\exp(i a \cosh(u))$  is an analytic function of  $u$  — the only essential singularities are in  $u = \pm\infty$  — the Laurent expansion (26) is well defined for any complex  $u$ .

If we now set  $z_1 = \frac{m}{\hbar} \sqrt{W} \frac{r_a^2}{\rho(t_a)}$ ,  $z_2 = \frac{m}{\hbar} \sqrt{W} \frac{r_b^2}{\rho(t_b)}$  (note that  $r_a^2; r_b^2 \geq 0$ ) and define  $\rho(t) = \sqrt{\sum_{i < j}^4 (\mathbf{u}_i(t) \wedge \mathbf{u}_j(t))^2}$  and  $b(t) = \exp\left(-i2 \arcsin \sqrt{\frac{V(t)}{\rho(t)}}\right)$  we can formulate the kernel in the desired form (23). The wave functions can be then identified as

$$\begin{aligned} \psi_{n,l}(r, u, t) = & \sqrt{\frac{1}{\pi}} \sqrt{\frac{n!}{\Gamma(n+l+1)}} \left( \sqrt{\frac{m}{\hbar \rho(t)}} W^{1/4} \right)^{l+1} \\ & \times [b(t)]^{n+\frac{l+1}{2}} r^l L_n^l \left( \frac{m}{\hbar \rho(t)} \sqrt{W} r^2 \right) \exp \left( \frac{m}{2\hbar} \left[ \frac{i}{2} \frac{\dot{\rho}(t)}{\rho(t)} - \frac{\sqrt{W}}{\rho(t)} \right] r^2 \right) \exp -l(u + \Gamma t - \frac{\beta}{2}) \end{aligned} \quad (28)$$

$$\begin{aligned} \psi_{n,l}^{(*)}(r, u, t) = & \sqrt{\frac{1}{\pi}} \sqrt{\frac{n!}{\Gamma(n+l+1)}} \left( \sqrt{\frac{m}{\hbar \rho(t)}} W^{1/4} \right)^{l+1} [b^*(t)]^{n+(l+1)/2} r^l L_n^l \left( \frac{m}{\hbar \rho(t)} \sqrt{W} r^2 \right) \\ & \times \exp \left( -\frac{m}{2\hbar} \left[ \frac{i}{2} \frac{\dot{\rho}(t)}{\rho(t)} + \frac{\sqrt{W}}{\rho(t)} \right] r^2 \right) \exp l \left( u + \Gamma t - \frac{\beta}{2} \right) \end{aligned} \quad (29)$$

We have used the usual convention that  $\psi_{n,l}(r, u, t) = \langle r, u | \psi_{n,l}(t) \rangle$ . From previous work, it is obvious that  $\psi_{n,l}^{(*)}(r, u, t)$  cannot be associated with  $\psi_{n,l}^*(r, u, t)$ ; in fact, we will see that  $\psi_{n,l}^{(*)}(r, u, t) = \psi_{n,l}(r, -u, -t)$ . It is also clear that neither  $\psi_{n,l}(r, u, t)$  nor  $\psi_{n,l}^{(*)}(r, u, t)$  belong to the ordinary Hilbert space because they cannot be normalized in the usual manner (they do not belong to the space of square integrable functions  $\ell^2$ ). This is in agreement with the work in refs. 1 and 4. Notice that the wave function (28) satisfies the time-dependent Schrödinger equation<sup>5</sup>

$$\left( i\hbar \frac{\partial}{\partial t_b} - \hat{H}(r_b, u_b) \right) \langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = 0, \quad t_b > t_a \quad (30)$$

where

$$\begin{aligned} \hat{H} = & \frac{1}{2m} \left[ \hat{p}_r^2 - \frac{1}{r^2} \hat{p}_u^2 + m^2 \Omega^2 r^2 \right] - \Gamma \hat{p}_u \\ = & \frac{1}{2m} \left[ -\hbar^2 \frac{\partial^2}{\partial r^2} - \frac{\hbar^2}{r} \frac{\partial}{\partial r} + \frac{\hbar^2}{r^2} \frac{\partial^2}{\partial u^2} + m^2 \Omega^2 r^2 \right] + i\hbar \Gamma \frac{\partial}{\partial u} \end{aligned} \quad (31)$$

is the Bateman Hamiltonian [1]. An interesting point is that the Feynman–Hibbs method uniquely prescribes that  $\hat{H}$  must be constructed in terms of the Laplace–Beltrami Laplacian;  $\Delta = \frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_u^2$  rather than from the canonical Laplacian;  $\Delta_{\text{can}} = (\partial_r + 1/2r)^2 + \frac{1}{r^2} \partial_u^2$ . Let us now define the *radial* kernel  $\langle r_b; t_b | r_a; t_a \rangle_{n,l}$  as [11]

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \sum_{n,l} \frac{\langle r_b; t_b | r_a; t_a \rangle_{n,l}}{\pi \sqrt{r_a r_b}} \exp l(\alpha(t) - \Delta u) \quad (32)$$

<sup>5</sup>On the other hand,  $\psi_{n,l}^{(*)}(r, u, t)$  fulfills the time-reversed (time-dependent) Schrödinger equation (45).

The corresponding wave function  $\psi_{n,l}(r, t) = \langle r | \psi_{n,l}(t) \rangle$  then reads

$$\begin{aligned} \psi_{n,l}(r, t) = & \sqrt{\frac{n!}{\Gamma(n+l+1)}} \left( \sqrt{\frac{m}{\hbar \rho(t)}} W^{1/4} \right)^{l+1} \\ & \times L_n^l \left( \frac{m \sqrt{W}}{\hbar \rho(t)} r^2 \right) [b(t)]^{n+(l+1)/2} r^{l+(1/2)} \exp \frac{m}{2\hbar} \left[ \frac{i \dot{\rho}(t)}{2 \rho(t)} - \frac{\sqrt{W}}{\rho(t)} \right] r^2 \end{aligned} \quad (33)$$

It is easy to check that  $\psi_{n,l}(r, t)$  fulfills both the orthonormalization condition and the resolution of unity

$$\int_{-\infty}^{\infty} dr \psi_{n,l}^{(*)}(r, t) \psi_{n',l}(r, t) = \delta_{nn'}, \quad \sum_{n=0}^{\infty} \psi_{n,l}^{(*)}(r, t) \psi_{n,l}(r', t) = \delta(r - r') \quad (34)$$

We also note that both the radial kernel and the wave function (33) satisfy the Schrödinger equation

$$\left( i\hbar \frac{\partial}{\partial t_b} - \hat{H}_l(r_b) \right) \langle r_b; t_b | r_a; t_a \rangle_{n,l} = 0, \quad t_b > t_a \quad (35)$$

where

$$\hat{H}_l = \frac{1}{2m} \left[ -\hbar^2 \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{r^2} \left( l^2 - \frac{1}{4} \right) + m^2 \Omega^2 r^2 \right] - i\hbar \dot{\alpha}(t) l - i\hbar \Gamma l \quad (36)$$

The term proportional to  $1/r^2$  is analogous to the *centrifugal barrier* and so the quantum number  $l$  can be viewed as an analog of the *azimuthal* quantum number. Note that, owing to the structure of  $\alpha(t_a, t_b)$ , the term  $\dot{\alpha}(t) + \Gamma$  must be zero.

Since the generalized Laguerre polynomials  $L_n^l$  are defined for all  $l \in \mathbb{C}$  indices,<sup>6</sup> the wave functions (33) satisfy the time-dependent Schrödinger equation with the Hamiltonian (36). Namely, when we continue  $l$  to the values  $\pm \frac{1}{2}$ , the Hamiltonian  $\hat{H}_l$  describes the one-dimensional LHO. If we make use of the rules connecting Hermite polynomials with the  $l = \pm \frac{1}{2}$  Laguerre polynomials [25], we may rewrite the continued radial wave functions in a simple form

$$\begin{aligned} \psi_{n,(1/2)}(r, t) = & \frac{1}{2^{2n+1}} \sqrt{\frac{1}{n! \Gamma(n + \frac{3}{2})}} \left( \sqrt{\frac{m}{\hbar \rho(t)}} W^{(1/4)} \right)^{1/2} \\ & \times H_{2n+1} \left( \sqrt{\frac{m}{\hbar \rho(t)}} W^{(1/4)} r \right) [b(t)]^{n+(3/4)} \exp \frac{m}{2\hbar} \left[ \frac{i \dot{\rho}(t)}{2 \rho(t)} - \frac{\sqrt{W}}{\rho(t)} \right] r^2 \end{aligned} \quad (37)$$

$$\begin{aligned} \psi_{n,-(1/2)}(r, t) = & \frac{1}{2^{2n}} \sqrt{\frac{1}{n! \Gamma(n + \frac{1}{2})}} \left( \sqrt{\frac{m}{\hbar \rho(t)}} W^{(1/4)} \right)^{1/2} \\ & \times H_{2n} \left( \sqrt{\frac{m}{\hbar \rho(t)}} W^{(1/4)} r \right) [b(t)]^{n+(1/4)} \exp \frac{m}{2\hbar} \left[ \frac{i \dot{\rho}(t)}{2 \rho(t)} - \frac{\sqrt{W}}{\rho(t)} \right] r^2 \end{aligned} \quad (38)$$

Finally, we note that the quantum numbers  $n$  and  $l$  appearing in (28) and (29) are not independent. So far the only restriction was that  $n \geq 0$  integers. Yet, for a consistent probabilistic interpretation (in the

<sup>6</sup>Analytic continuation for  $l$  with  $\Re(l) = -1, -2, -3, \dots$  is, however, required, see for example, ref. 26.

$r$  variable) and for analytical continuation in (37) and (38) the wave function  $\psi_{n,l}(r, u, t)$  is required to be bounded for  $|r| < \infty$ . Using the asymptotic expansion for  $L_n^l(z)$  (see, for example, refs. 14 and 25)

$$L_n^l(z) = \frac{[\Gamma(n+l+1)]^2}{n!\Gamma(l+1)} {}_1F_1(-n, l+1; z) \begin{cases} \approx \frac{\Gamma(n+l+1)}{n!} (-z)^n, & z \rightarrow \infty \\ \approx 1 - \frac{n!}{l!} z, & z \rightarrow 0 \end{cases}$$

( ${}_1F_1(-n, l+1; z)$  is the confluent hypergeometric series [14]), it is not difficult to see that the only allowed values of  $n$  and  $l$  at which  $\psi_{n,l}(r, u, t)$  fulfills the above requirements are those for which

$$2n + l + 1 = 0, \pm 1, \pm 2, \dots, \quad l \neq 0 \quad (39)$$

Thus  $|l| \geq 1$  and  $n \geq 0$ .

#### 4. Other features of the quantum Bateman system

Let us now examine the meaning of the quantum numbers  $n$  and  $l$ . To this end, we study the algebraic structure of the Hamiltonian (31). In refs. 1 and 4 the following ladder operators were introduced:

$$A = \frac{1}{\sqrt{2\hbar m \Omega}} [\hat{p}_1 - im\Omega x_1], \quad B = \frac{1}{\sqrt{2\hbar m \Omega}} [\hat{p}_2 - im\Omega x_2] \quad (40)$$

with  $[A, A^\dagger] = [B, B^\dagger] = 1$ ,  $[A, B] = [A, B^\dagger] = 0$ . Then the Hamiltonian (31) can be written as

$$\hat{H} = \hbar\Omega(A^\dagger A - B^\dagger B) + i\hbar\Gamma(A^\dagger B^\dagger - AB) = 2\hbar(\Omega C - \Gamma J_2) \quad (41)$$

where we have made explicit the associated  $SO(2, 1) \equiv SU(1, 1)$  algebraic structure

$$J_+ = A^\dagger B^\dagger, \quad J_- = AB, \quad J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1), \quad [J_+, J_-] = -2J_3, \quad [J_3, J_\pm] = \pm J_\pm \quad (42)$$

Note that  $C = \frac{1}{2}(A^\dagger A - B^\dagger B)$  is the  $SU(1, 1)$  Casimir operator and  $J_2 = -i(J_+ - J_-)/2$ . Using (40) it is now simple to check that

$$C = \frac{1}{4\hbar\Omega m} \left[ \hat{p}_r^2 - \frac{1}{r^2} \hat{p}_u^2 + m^2 \Omega^2 r^2 \right], \quad J_2 = \frac{1}{2\hbar} \hat{p}_u \quad (43)$$

We now turn to determine the behavior of Bateman's system under time reversal. It is indeed important to recognize that the system described by the Hamiltonian (41) is both conservative and invariant under time reversal, contrary to what is stated sometimes in the literature [1].

In ref. 15, it was pointed out that a time-reversal transformation must be formulated in terms of kinematic variables. This means particularly that the admissible time-reversal transformation must be consistent with the algebraic structure of the operators representing the (kinematic) observables and that in the absence of forces the dynamic equations must be left invariant. From this it may be deduced [15] that

$$\mathcal{T}C\mathcal{T}^{-1} = C, \quad \mathcal{T}J_2\mathcal{T}^{-1} = J_2 \quad (44)$$

and so  $\mathcal{T}\hat{H}\mathcal{T}^{-1} = \hat{H}$ . The latter is not compatible with the time reversal presented in ref. 1.

It should be stressed that properties (44) come from an algebraic structure of  $SU(1, 1)$  and as such are independent of a particular representation. On the other hand, the peculiar behavior of  $J_2$  under “ $\dagger$ ” is representation dependent. Indeed, from its definition,  $J_2$  appears to be Hermitian, but (29) implies that it has a purely imaginary spectrum in  $|\psi_{n,l}(t)\rangle$  (and this holds for all  $t$ ). The root of this “pathological” behavior is in the nonexistence of a unitary irreducible representation of  $SU(1, 1)$



in which  $J_2$  would have at the same time a real and discrete spectrum [27]. Nevertheless, both the discreteness and complexness of the  $J_2$  spectrum are vital in our analysis since they bring dissipative features in the dynamics (see also refs. 1, 3, and 4). Thus, the usual unitary representations of  $SU(1, 1)$  are clearly not useful for our purpose. On the other hand, by resorting to nonunitary representations of  $SU(1, 1)$  (known as a non-unitary principal series [28, 29]), we lose the Hermiticity of  $J_2$  and hence the spectral theorem along with the resolution of unity.

The reconciliation can be found in redefining the inner product [4, 29] to get a unitary irreducible representation (known as complementary series [28, 29]) out of non-unitary principal series. This may be done noticing (using (23), (29), and Schwinger's prescription (4)) that the states  $\psi_{n,l}^{(*)}$  are not simple complex conjugates of  $\psi_{n,l}$  because they fulfill the time-dependent Schrödinger equation

$$\left( i\hbar \frac{\partial}{\partial t} + \mathcal{T} \hat{H}^\dagger(r, u) \mathcal{T}^{-1} \right) \psi_{n,l}^{(*)}(r, u, t) = 0 \quad (45)$$

with the (effectively) non-Hermitian Hamiltonian.<sup>7</sup> Accordingly, what we have loosely denoted in (23) as  $\psi_{n,l}^{(*)}(r, u, t)$  is really

$$\langle \mathcal{T} \psi_{n,l}(-t) | r, u \rangle = \langle \psi_{n,l}(-t) | r, -u \rangle^* = \psi_{n,l}(r, -u, -t) \quad (46)$$

as can be double-checked from the explicit form (29). For the sake of simplicity, we use  $[\mathcal{T} |\psi_{n,l}(t)\rangle]^\dagger = \langle \mathcal{T} \psi_{n,l}(t) |$ . Clearly, if  $J_2$  were Hermitian then  $\psi_{n,l}^{(*)} = \psi_{n,l}^*$  as one would expect.

The above considerations have some important implications. To see this, let us rewrite the kernel by means of the states  $|\psi_{n,l}(t)\rangle$

$$\begin{aligned} \langle r_b, u_b; t_b | r_a, u_a; t_a \rangle &= \sum_{n,l} \psi_{n,l}(r_b, u_b, t_b) \psi_{n,l}^{(*)}(r_a, u_a, t_a) \\ &= \sum_{n,l} \langle r_b, u_b | \psi_{n,l}(t_b) \rangle \langle \mathcal{T} \psi_{n,l}(-t_a) | r_a, u_a \rangle \end{aligned} \quad (47)$$

We can formally introduce the conjugation operation (“bra vector”) as  $\langle \psi_{n,l}(t) | \equiv [\mathcal{T} |\psi_{n,l}(-t)\rangle]^\dagger$ . Then the resolution of unity is written in a simple form

$$\sum_{n,l} |\psi_{n,l}(t)\rangle \langle \psi_{n,l}(t) | = 1 \quad (48)$$

The price that has been paid for this simplicity is that we have endowed the Hilbert space with a new inner product. Note that under this product  $|\psi_{n,l}(t)\rangle$  has a finite (and positive) norm and that  $J_2$  is Hermitian with respect to the new inner product. Indeed, integrating by parts we get

$$\begin{aligned} \langle \psi_{n,l}(t) | J_2 \psi_{m,k}(t) \rangle &= \int dr du r \psi_{n,l}^{(*)}(r, u, t) \left( -\frac{i}{2} \frac{\partial}{\partial u} \right) \psi_{m,k}(r, u, t) \\ &= \int dr du r \left[ \left( -\frac{i}{2} \frac{\partial}{\partial u} \right) \psi_{n,l}(r, u, t) \right]^{(*)} \psi_{m,k}(r, u, t) \\ &= \langle J_2 \psi_{n,l}(t) | \psi_{m,k}(t) \rangle \end{aligned} \quad (49)$$

<sup>7</sup>It should be realized that the Hermiticity (in contrast with self-adjointness) is dependent on the properties of the Hilbert space in question. So, for instance, while  $\hat{H}$  is Hermitian in the space of square integrable functions  $\ell^2$  this is not so in the Hilbert space spanned by (not normalizable) functions  $\psi_{n,l}(r, u, t)$ .

In (49) we have applied (44) and (46) together with the fact that the “surface” term is zero (if  $k \neq l$  then integration with respect to the variable  $r$  gives zero [25], if  $k = l$  then the product  $\psi_{n,l}^{(*)} \psi_{m,l}$  is  $u$  independent).

Note that (49) implies that  $[\mathcal{T} J_2 \mathcal{T}^{-1}]^\dagger = J_2$ , or equivalently,  $\mathcal{T} J_2^\dagger \mathcal{T}^{-1} = J_2$ . Indeed

$$\langle J_2 \psi_{n,l}(t) | = [\mathcal{T} J_2 |\psi_{n,l}(-t)\rangle]^\dagger = \langle \psi_{n,l}(t) | [\mathcal{T} J_2 \mathcal{T}^{-1}]^\dagger \quad (50)$$

The spurious time irreversibility of  $J_2$  apparent in (50) is an obvious consequence of dealing with the non-unitary representation of  $SU(1, 1)$ . From a mathematical point of view, we can interpret the relation  $\mathcal{T} J_2 \mathcal{T}^{-1} = J_2^\dagger$  as a self-adjoint extension of  $J_2$  in the space spanned by the  $|\psi_{n,l}(t)\rangle$  vectors. The key observation then is that

$$\langle \psi_{n,l}(t) | = [\mathcal{T} |\psi_{n,l}(-t)\rangle]^\dagger = \left[ \mathcal{T} e^{(it/\hbar) \hat{H}} |\psi_{n,l}(0)\rangle \right]^\dagger = \langle \psi_{n,l}(0) | \exp\left(\frac{it}{\hbar} \hat{H}\right) \quad (51)$$

Thus, the time-evolution operator is unitary under the new inner product. It is this unitarity condition, intrinsically built in the kernel formula (6) (and successively taken over by the Feynman–Hilbert prescription), which naturally leads to a “consistent” inner product introduced in a somehow intuitive manner in refs. 1 and 4. From now on the modified inner product will be always tacitly assumed.

So far, we have dealt with the peculiar structure of the Hilbert space. To interpret the quantum numbers  $n, l$  labeling the constituent states, we start with the observation that from the explicit form (29) one can readily construct the Hermitian operator  $\tilde{C}$  (commuting with  $J_2$ ) which is diagonalized by  $\psi_{n,l}(r, u, t)$ . Indeed one may check that

$$J_2 \psi_{n,l}(r, u, t) = -\frac{i}{2} \frac{\partial}{\partial u} \psi_{n,l}(r, u, t) = i \frac{l}{2} \psi_{n,l}(r, u, t) \quad (52)$$

$$\tilde{C} \psi_{n,l}(r, u, t) = \frac{\sqrt{W}}{2\Omega\rho} (2n + l + 1) \psi_{n,l}(r, u, t) \quad (53)$$

Here

$$\tilde{C} = C - \frac{m}{4\Omega\hbar} \left( \Omega^2 - \frac{W}{\rho^2} - \frac{\dot{\rho}^2}{4\rho^2} \right) r^2 + \frac{i}{4\Omega\rho} \dot{\rho} \left( r \frac{\partial}{\partial r} + 1 \right) = e^{2\zeta} \hat{R}(t) C \hat{R}^{-1}(t) \quad (54)$$

The unitary operator  $\hat{R}(t)$  has the form

$$\hat{R}(t) = \hat{S}(\xi; t) \exp(i\zeta G_A) \exp(i\zeta G_B), \quad \hat{S}(\xi; t) = \exp\left(i \xi r^2\right) \quad (55)$$

$$G_A = i \frac{1}{2} \left( A^2 - (A^\dagger)^2 \right), \quad G_B = i \frac{1}{2} \left( B^2 - (B^\dagger)^2 \right) \quad (56)$$

$$\zeta = \frac{1}{4} \ln \left( \frac{W}{\Omega^2 \rho^2} \right), \quad \xi = \frac{m}{4\hbar} \frac{\dot{\rho}}{\rho} \Rightarrow \xi = -\frac{m}{2\hbar} \dot{\zeta} \quad (57)$$

Note that  $G_A$  and  $G_B$  are the  $SU(1, 1)$  displacement operators (i.e., generalized coherent states generators) [28]. The connection between  $SU(1, 1)$  squeezed states and damped oscillators was firstly proposed in ref. 30.

From (54), we find that  $\tilde{C}$  is Hermitian whenever  $\rho$  is a real function and  $\mathcal{T} \tilde{C}(t) \mathcal{T}^{-1} = \tilde{C}(-t)$ . The latter together with (53), in turn, implies that

$$\mathcal{T} |\psi_{n,l}(t)\rangle = |\psi_{n+l, -l}(-t)\rangle \quad (58)$$

and so, namely, in the static case (i.e., when  $\rho(t) = \text{const}$  and  $V(t) = \rho \sin^2(\Omega t)$ , see also Sect. 6) we have

$$J_2 |\psi_{n,l}^s\rangle = i \frac{l}{2} |\psi_{n,l}^s\rangle, \quad C |\psi_{n,l}^s\rangle = \frac{1}{2} (2n + l + 1) |\psi_{n,l}^s\rangle \quad (59)$$

where we used  $|\psi_{n,l}^s\rangle \equiv |\psi_{n,l}^s(0)\rangle$ . Using (53), (54), and (59), we can find the relation between  $|\psi_{n,l}(t)\rangle$  and the stationary states  $|\psi_{n,l}^s\rangle$

$$|\psi_{n,l}(t)\rangle = \hat{R}(t) |\psi_{n,l}^s\rangle \quad (60)$$

Thus, vectors  $|\psi_{n,l}(t)\rangle$  appearing in the spectral decomposition of the kernel (23) have a tight connection with  $SU(1, 1)$  coherent states. In fact, it may be shown that they describe coherent states which (if expressed in the  $|r, u\rangle$  representation) rotate in their position spread and (or) pulsate in their width (“breathers”) [15].

## 5. Pancharatnam phase for Bateman’s dual system

Having obtained, in Sect. 3, the time-dependent wave functions for our system, we are now ready to study the corresponding geometric phases. In this connection, we remind the reader that a Hilbert space  $\mathcal{H}$  is a line bundle over the projective space  $\mathcal{P}$ , i.e., the equivalence class of all vectors that differ by a multiplication with a complex number. We denote a generic element of  $\mathcal{P}$  as  $|\tilde{\psi}\rangle$ . The inner product on  $\mathcal{H}$  naturally endows  $\mathcal{P}$  with two important geometric structures: the Fubini–Study metric [31, 32] and a  $U(1)$  connection (Berry connection [32, 33])

$$\mathcal{A} = i \langle \tilde{\psi} | d | \tilde{\psi} \rangle \quad (61)$$

When a point evolves in  $\mathcal{P}$  along a closed loop, say  $\gamma$ , the total phase change  $\phi_{\text{tot}}$  of  $|\psi\rangle$  in  $\mathcal{H}$  consists of two contributions: the dynamical part (with  $\tau$  being the time period at which the system traverses  $\gamma$ )

$$\phi_{\text{dyn}} = -\hbar^{-1} \int_0^\tau dt \langle \psi(t) | \hat{H} | \psi(t) \rangle \quad (62)$$

and the geometric part (Berry phase)

$$\begin{aligned} \exp(i\phi_B) &\equiv \exp(i\phi_{\text{tot}} - i\phi_{\text{dyn}}) = \langle \psi(0) | \psi(\tau) \rangle \exp\left(i \int_0^\tau dt \langle \psi(t) | i \frac{d}{dt} | \psi(t) \rangle\right) \\ &= \exp\left(i \int_0^\tau dt \langle \tilde{\psi}(t) | i \frac{d}{dt} | \tilde{\psi}(t) \rangle\right) = \exp\left(i \int_\gamma \mathcal{A}\right) \end{aligned} \quad (63)$$

So the Berry phase  $\phi_B$  may be geometrically understood as anholonomy with respect to Berry’s connection in the projective space  $\mathcal{P}$  [31]. The above relations are meant only for states  $|\psi(t)\rangle$  that are normalized to unity. In general, corresponding division of the state norm must be invoked. For simplicity we omit this in the following.

The geometric phase can also be defined for open paths (Pancharatnam’s phase) [34]. The trick is that any open path in  $\mathcal{P}$  can be closed with a (Fubini–Study) geodesic. The geometric phase of such a loop is then defined to be equal to the geometric phase associated with the open path. This is so, because parallel transport along a geodesic does not bring any anholonomy. So, if  $\gamma_o$  is an open path on  $\mathcal{P}$  and  $\gamma_g$  is the corresponding geodesic on  $\mathcal{P}$ , then the associated Pancharatnam phase  $\phi_P$  reads

$$\begin{aligned} \exp(i\phi_P) &= \exp\left(i \int_{\gamma_o + \gamma_g} \mathcal{A}\right) = \exp\left(i \int_{\gamma_o} \mathcal{A}\right) \\ &= \exp\left(i \int_{t_i}^{t_f} dt \langle \tilde{\psi}(t) | i \frac{d}{dt} | \tilde{\psi}(t) \rangle\right) = \langle \psi(t_i) | \psi(t_f) \rangle \exp\left(i \int_{t_i}^{t_f} dt \langle \psi(t) | i \frac{d}{dt} | \psi(t) \rangle\right) \end{aligned} \quad (64)$$

Note that  $\phi_P$  is well defined only if the endpoints are not orthogonal. It should also be clear that both  $\phi_B$  and  $\phi_P$  are defined only modulo  $2\pi$ .

To determine the geometric phase for Bateman's system we first compute the dynamic phase. Using (29), (53), and (54) we find that

$$\begin{aligned}\phi_{\text{dyn}} &= -\hbar^{-1} \int_{t_i}^{t_f} dt \langle \psi_{n,l}(t) | \hat{H} | \psi_{n,l}(t) \rangle \\ &= -(2n+l+1) \int_{t_i}^{t_f} dt \frac{\sqrt{W}}{2\rho} \left( \frac{\dot{\rho}^2}{4W} + \frac{\Omega^2 \rho^2}{W} + 1 \right) + i \Gamma l (t_f - t_i)\end{aligned}\quad (65)$$

To proceed further, it is convenient to define the complex number  $z(t) = i \sqrt{V(t)/\rho(t)} + \sqrt{1 - V(t)/\rho(t)}$  with  $z\bar{z} = 1$ . When evolving in the time interval  $(t_i, t_f)$ ,  $z(t)$  traverses a curve  $\gamma$  in the Gaussian plane. The index of the curve  $\gamma$  (i.e., the number of revolutions around the origin) is then defined as [35]

$$\text{ind } \gamma = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z}\quad (66)$$

Accordingly, the dynamic phase can be written as

$$\phi_{\text{dyn}} = -(2n+l+1) \int_{t_i}^{t_f} dt \left( \frac{\dot{\rho}^2}{8\rho\sqrt{W}} + \frac{\Omega^2 \rho}{2\sqrt{W}} \right) - (2n+l+1)(\pi \text{ind } \gamma) + i \Gamma l (t_f - t_i)\quad (67)$$

Using (66) we may write also  $\phi_{\text{tot}}$  in a simple form

$$\begin{aligned}\phi_{\text{tot}} &= \arg \{ \langle \psi_{n,l}(t_i) | \psi_{n,l}(t_f) \rangle \} \\ &= -(2n+l+1) \left( \arcsin \sqrt{\frac{V(t_f)}{\rho(t_f)}} - \arcsin \sqrt{\frac{V(t_i)}{\rho(t_i)}} \right) - \frac{\pi}{2} n_{i,f} + i \Gamma l (t_f - t_i) \\ &= -(2n+l+1)(2\pi \text{ind } \gamma) - \frac{\pi}{2} n_{i,f} + i \Gamma l (t_f - t_i)\end{aligned}\quad (68)$$

Here  $n_{i,f}$  is the Morse index of the classical trajectory running between  $\mathbf{x}_i$  and  $\mathbf{x}_f$ .

The fact that we get an imaginary piece both in  $\phi_{\text{tot}}$  and  $\phi_{\text{dyn}}$  should not be surprising because we work with the modified inner product. Notice that the "troublesome" contribution in  $\phi_{\text{tot}}$  and  $\phi_{\text{dyn}}$  correctly flips the sign if one passes from  $\psi(\dots)$  to  $\psi^{(*)}(\dots)$ .

Substituting (67) and (68) into (64), we finally obtain Pancharatnam's phase as

$$\phi_P = (2n+l+1) \int_{t_i}^{t_f} dt \left( \frac{\dot{\rho}^2}{8\rho\sqrt{W}} + \frac{\Omega^2 \rho}{2\sqrt{W}} \right) - (2n+l+1)(\pi \text{ind } \gamma) - \frac{\pi}{2} n_{i,f}\quad (69)$$

Note that because  $\rho$ ,  $W$ , and  $V$  are constructed only from solutions of the classical equations of motion,  $\phi_P$  is manifestly  $\hbar$  independent.

In ref. 15, we show that the first three contributions in (69) correspond to overall ground-state fluctuations of  $\hat{p}$  and  $\hat{x}$  gathered during the time period  $t_f - t_i$ . While these are basic characteristics of the ground-state wave packet, the Morse index, on the other hand, reflects the geometrical features of the path traversed by the ground-state wave packet in configuration space.

## 6. The ground state of one-dimensional LHO

In this section, we consider the reduction of Bateman's system to the one-dimensional LHO. This is of interest in view of the analysis of ref. 8, where it has been shown that (1) the Bateman system

belongs to the class of deterministic quantum systems proposed by 't Hooft [7], and (2) it reduces to the truly quantum one-dimensional LHO, under an appropriate constraint condition on the relevant degrees of freedom.

By using the explicit form of the wave functions (29) and (38), we can easily see that the following relation between the one-dimensional LHO wave function  $\psi_n^{\text{lho}}(r, t) \equiv \psi_{n, -(1/2)}(r, t)$  and Bateman's system wave function  $\psi_{n,l}(r, u, t)$  holds

$$\psi_n^{\text{lho}}(r, t) = \sqrt{\pi r} \psi_{n, -(1/2)}\left(r, -\Gamma t + \frac{\beta}{2}, t\right) \quad (70)$$

Let us now consider the geometric phase of the one-dimensional LHO "inherited" through the reduction of (70). To do this, we assume that  $\tau$  is the mutual period of both  $\rho$  and  $V$ , then the following chain of equalities holds:

$$\begin{aligned} \psi_n^{\text{lho}}(r, \tau) &= \sqrt{\pi r} \psi_{n, -(1/2)}(r, u, \tau)|_{u=\beta/2-\tau\Gamma} = \sqrt{\pi r} \left( \exp(i\phi_{\text{tot}}) \psi_{n, -(1/2)}(r, u, 0) \right)|_{u=\beta/2-\tau\Gamma} \\ &= \sqrt{\pi r} \left( \exp \left\{ i \left[ \phi_{\text{tot}} - \frac{\tau \Gamma p u}{\hbar} \right] \right\} \psi_{n, -(1/2)}(r, u + \tau\Gamma, 0) \right)|_{u=\beta/2-\tau\Gamma} \\ &= \exp \left\{ i \left[ \phi_B + \frac{1}{\hbar} \int_0^\tau \langle \psi_n^{\text{lho}}(t) | \hat{H}_{-(1/2)} | \psi_n^{\text{lho}}(t) \rangle dt \right] \right\} \psi_n^{\text{lho}}(r, 0) \end{aligned} \quad (71)$$

Now, because both  $\rho$  and  $V$  are, by assumption, periodic with period  $\tau$ , the wave function  $\psi_n^{\text{lho}}(r, t)$  must be  $\tau$  periodic as well, and thus  $\psi_n^{\text{lho}}(r, 0) = e^{i2\pi m} \psi_n^{\text{lho}}(r, \tau)$ , where  $m$  is an arbitrary integer. Using the fact that  $\phi_B$  is defined modulo  $2\pi$  we can write

$$\int_0^\tau \langle \psi_n^{\text{lho}}(t) | \hat{H}_{-(1/2)} | \psi_n^{\text{lho}}(t) \rangle dt = \hbar (2\pi n - \phi_B) \quad (72)$$

In the case when  $\psi_n^{\text{lho}}(r, t)$  are eigenstates of  $\hat{H}_{-(1/2)}$  we obtain that the energy spectrum quantizes in the following way:

$$E_n^{\text{lho}} = \frac{\hbar}{\tau} (2\pi n - \phi_{AB}) \quad (73)$$

The foregoing analysis can be easily extended to the case  $l = +(1/2)$ .

To practically realize the above scheme, let us consider an explicit example in which the following fundamental system of solutions is chosen:

$$\begin{aligned} u_1^1(t) = -u_2^2(t) &= \sqrt{2} \cos(\Omega t) \cosh(\Gamma t), & u_1^2(t) = -u_2^1(t) &= \sqrt{2} \cos(\Omega t) \sinh(\Gamma t) \\ v_1^1(t) = -v_2^2(t) &= \sqrt{2} \sin(\Omega t) \cosh(\Gamma t), & v_1^2(t) = -v_2^1(t) &= \sqrt{2} \sin(\Omega t) \sinh(\Gamma t) \end{aligned} \quad (74)$$

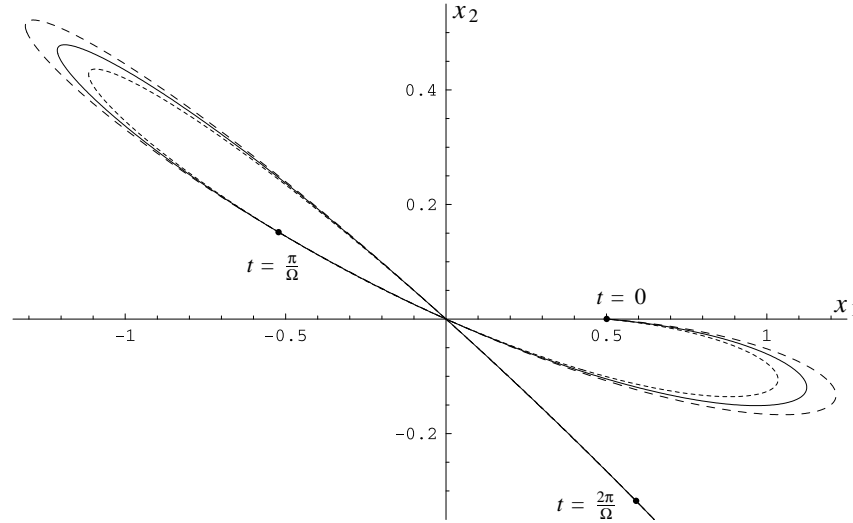
The Wronskian is  $W = 4\Omega^2$  and the determinant  $D = 4 \sin^2 \Omega(t_b - t_a)$ . As a result, we obtain for the classical action and the fluctuation factor the following realizations:

$$S_{\text{cl}} = \frac{m\Omega}{2 \sin[\Omega(t_b - t_a)]} \left\{ (r_a^2 + r_b^2) \cos[\Omega(t_b - t_a)] - 2r_a r_b \cosh[u_b - u_a - \Gamma(t_b - t_a)] \right\} \quad (75)$$

$$F[t_a, t_b] = \frac{m}{2\pi \hbar} \frac{\Omega}{|\sin[\Omega(t_b - t_a)]|} \quad (76)$$

The kernel (without the Morse-index contribution) is obtained as  $\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = F[t_a, t_b] e^{iS_{\text{cl}}}$ . It is easy to check that it satisfies the time-dependent Schrödinger equation (31). Note also that the Schrödinger equation alone is unable to indicate the presence of the Morse index.

**Fig. 1.** Orbits in the  $(x_1, x_2)$  configuration-space orbits. Focal points occur in  $x_a$  and  $x_b$  at times  $t = \pi/\Omega$  and  $t = 2\pi/\Omega$ , respectively. Irrespective of the initial-time momenta, all orbits reach the focal points at the same time.



Rewriting the kernel; applying expansions (26) and (27); and using  $V(t) = 2 \sin^2(\Omega t)$ ,  $\rho(t) = 2$  and  $b(t) = e^{-2i\Omega t}$ , we obtain, see ref. 15, both the wave function and the radial wave functions

$$\psi_{n,l}(r, u, t) = \sqrt{\frac{n!}{\pi \Gamma(n+l+1)}} \left(\frac{m\sqrt{\Omega}}{\hbar}\right)^{l+(1/2)} r^l \exp\left(-\frac{m\Omega}{2\hbar} r^2\right) L_n^l\left(\frac{m\Omega}{\hbar} r^2\right) \times \exp[-l(u + \Gamma t)] \exp[-i\Omega(2n+l+1)t] \quad (77)$$

$$\psi_{n,l}(r, t) = \sqrt{\frac{n!}{\pi \Gamma(n+l+1)}} \left(\frac{m\sqrt{\Omega}}{\hbar}\right)^{l+(1/2)} r^{l+(1/2)} \exp\left(-\frac{m\Omega}{2\hbar} r^2\right) L_n^l\left(\frac{m\Omega}{\hbar} r^2\right) \times \exp[-i\Omega(2n+l+1)t] \quad (78)$$

Notice that  $\psi_{n,l}(r, t)$  is an eigenstate of  $\hat{H}_l$ .

Since  $V$  is periodic with *fundamental period*  $\tau = \pi/\Omega$  and because  $\psi_{n,-(1/2)}$  is an eigenstate of  $\hat{H}_{-(1/2)}$ , the energy spectrum of the related one-dimensional LHO is done by (73). The corresponding ground-state energy can be calculated from (69) and (73)

$$E_0^{\text{LHO}} = -\hbar \frac{\phi_B}{\tau} = \hbar \pi \frac{n_{a,b}}{\tau} \quad (79)$$

Thus, the fundamental system (74) reflects the underlying Bateman dual system in  $E_0^{\text{LHO}}$  only via the Morse index  $n_{a,b}$ . To find  $n_{a,b}$ , we may apply the Lagrangian manifold formalism [20, 24].

To get the Morse index appearing in (79), we notice that during the interval  $(0, \tau)$  orbits pass one caustic at the conjugate time  $\tau = \pi/\Omega$ , with multiplicity 2. The Morse index is then  $n_{a,b} = 2$  and, therefore, the ground state of the one-dimensional LHO obtained after reduction is  $E_0^{\text{LHO}} = \hbar\Omega$  (and not  $\hbar\Omega/2!$ ), see Fig. 1.

Actually the above analysis is still not the whole story since there is yet another contribution to the ground-state energy, reflecting the dissipative nature of the system when working with the  $SU(1, 1)$  non-unitary representation. Such a contribution manifests itself in the form of an additional phase factor

— the “dissipative” phase — and can be obtained by integrating over a closed-time loop [8, 36]. In the present approach, only the forward-in-time evolution was considered and the usual definition of geometric phase was used to obtain the above result. It is definitely a challenging task to extend the notion of geometric phases into dissipative systems, and work in this direction is currently in progress.

## 7. Conclusions

In this paper, we have studied the quantization of Bateman’s dual system of damped–antidamped harmonic oscillators [1, 2] by use of the Feynman–Hibbs kernel formula. This approach has several advantages with respect to the usual canonical formalism [1, 3, 4], in particular the fact that for quadratic systems, the kernel is *fully* expressible in terms of the fundamental system of solutions of classical equations of motion and is independent of the choice of such solutions. As a result, we have been able to construct the wave functions of the Bateman system purely from the equation of motion alone. Perhaps the most important point is that we did not need to invoke any Lagrangian (or Hamiltonian) formalism, and so the procedure naturally suits a quantization of dissipative systems.

Due to the nature of our approach, we were able to address some subtle issues connected with the quantization of Bateman’s system, namely, the effective non-Hermiticity of the Hamiltonian and the oddness of  $J_2$  under time reversal. We have found that the root of both these “pathologies” is in the use of the non-unitary irreducible representation of the  $SU(1, 1)$  dynamic group. The latter is, however, a prerequisite if one is interested in the dissipative features of the system. It was then necessary to redefine the inner product of the Hilbert space in such a way that the time evolution appeared to be unitary. We have then calculated the Pancharatnam geometric phase associated with the evolution of the system.

The results obtained have been used to probe the reduction of the two-dimensional Bateman system to the one-dimensional linear harmonic oscillator. Such a procedure has been investigated recently [8] (although from a different standpoint) in connection with ’t Hooft’s theory on deterministic Quantum Mechanics [7]. We have shown explicitly how the reduced (or radial) wave functions fulfill the time-dependent Schrödinger equation for the one-dimensional harmonic oscillator, when the “azimuthal” quantum number  $l$  is set equal to  $\pm(1/2)$ .

Similarly as in ref. 8, we have shown here that such a reduction equips the energy spectrum with the irreducible ground-state contribution that originates from the geometry of the path traversed by the system (either in space or time) rather than from the Hamiltonian alone. We have demonstrated that the geometric phase of the one-dimensional harmonic oscillator obtained via such a reduction bears a memory of the classical motion of the original Bateman dual system. This shadow of the underlying two-dimensional system reflects into the ground-state energy of the one-dimensional LHO, which is controlled by the Morse index affiliated with Bateman’s dual system.

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